MATHEMATICS

A Textbook for Class XII

PART II

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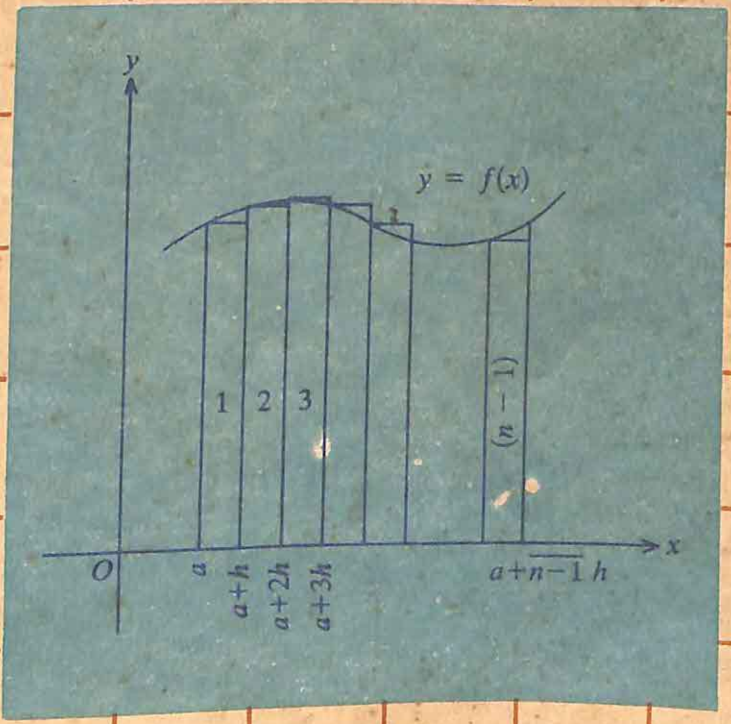

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MATHEMATICS

A Textbook for Class XII

Part II

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SCIENCE RELATED VALUES

Curiosity, quest for knowledge, objectivity, honesty and truthfulness, courage to question, systematic reasoning, acceptance after proof/verification, open-mindedness, search for perfection and team spirit are some of the basic values related to science. The processes of science, which help in searching the truth about nature and its phenomena are characterised by these values. Science aims at explaining things and events. Therefore to learn and practise science :

- **Be inquisitive about things and events around you.**
- **Have the courage to question beliefs and practices.**
- **Ask 'what', 'how' and 'why' and find your answers by critically observing, experimenting, consulting, discussing and reasoning.**
- **Record honestly your observations and experimental results in your laboratory or outside it.**
- **Repeat experiments carefully and systematically if required, but do not manipulate your results under any circumstance.**
- **Be guided by facts, reasons and logic. Do not be biased in one way or the other.**
- **Aspire to make new discoveries and inventions by sustained and dedicated work.**

Contents

6. Indefinite Integral **267**
6.1 Integral as Anti-Derivative 267
6.2 Integration by Substitution 278
6.3 Integration using Trigonometric Identities 283
6.4 Some Special Integrals 286
6.5 Integration through Partial Fractions 298
6.6 Integration by Parts 308
7.7 Integrals of Some More Types 314
6.8 Integration by Successive Reduction 323

7. Definite Integral **332**
7.1 Definite Integral as a Limit of a Sum 332
7.2 Fundamental Theorem of Calculus 347
7.3 Evaluation of Definite Integrals by Substitution 358
7.4 Some Properties of Definite Integrals 361
7.5 Areas of Bounded Regions 377

8. Differential Equations **386**
8.1 Introduction 386
8.2 Differential Equations: Order and Degree 386
8.3 Formation of Differential Equations 388
8.4 Solution of a Differential Equation 393
8.5 Equations with Variables Separable 394
8.6 Homogeneous Equations 399
8.7 First Order Linear Equation 401

9. Vector Algebra **406**
9.1 Vectors 406
9.2 Operations with Vectors 415
9.3 Products of Two Vectors 442
9.4 Moment of Force about a Point 476
9.5 Moment of a Couple 478
9.6 Triple Products 480

10. Three-dimensional Geometry **495**
10.1 Three-dimensional Space: Geometric Concepts 495
10.2 Straight Line in Space 505

10.3 Plane 512
10.4 Sphere 524

ANSWERS 529

Chapters 1-5 have been covered in Part I. Part III of this textbook will cover Chapters 11-14 .

CHAPTER 6

Indefinite Integral

IN CHAPTERs 3 and 4 we studied differentiation and its applications. In the next two chapters, we shall study another important process called integration. This also has several applications in science and technology as well as in other fields like economics.

Integration is the inverse process of differentiation. The process of finding $f(x)$, when its derivative $f'(x)$ is given, is known as integration. In this chapter, we study some methods of integration.

6.1 Integral as Anti-Derivative

To explain the meaning of the word 'integral', we first consider an example.

We know $$\frac{d}{dx}(\sin x) = \cos x.$$

The function $\cos x$ is the derivative of the function $\sin x$. On the other hand, we say that the function $\sin x$ is an integral of $\cos x$.

More generally, if $$\frac{d}{dx}(f(x)) = g(x),$$

we say that $f(x)$ is an integral of $g(x)$.

In other words, $f(x)$ is an integral of $g(x)$ when and only when $g(x)$ is the derivative of $f(x)$. For this reason, integral is sometimes described as 'anti-derivative'. It is also called 'indefinite integral' as opposed to the 'definite integral' which you will study in Chapter 7.

Notation

If $f(x)$ is an integral of $g(x)$, we write

$$\int g(x)\, dx = f(x)$$

and read the symbol $\int$ as 'integral'. In this notation dx separately does not have a meaning. This symbol dx conveys that x is the variable of integration; this means that the function $g(x)$, as a function of x, is taken, and we are finding a function $f(x)$ whose derivative (with respect to x) is $g(x)$.

We already know the formulas for the derivatives of many important functions. From these formulas, we can write down immediately the corresponding formulas for integrals, as shown in the table below:

S.No	*We already know the formula*	*Correspondingly one integral is given by the formula*
1.	$\frac{d}{dx}(x^n) = nx^{n-1}$ for all integers $n \neq 0$	$\int nx^{n-1}\,dx = x^n$ for all integers $n \neq 0$
2.	$\frac{d}{dx}(e^x) = e^x$	$\int e^x\,dx = e^x$
3.	$\frac{d}{dx}(\sin x) = \cos x$	$\int \cos x\,dx = \sin x$
4.	$\frac{d}{dx}(\cos x) = -\sin x$	$\int (-\sin x)\,dx = \cos x$
5.	$\frac{d}{dx}(\tan x) = \sec^2 x$	$\int \sec^2 x\,dx = \tan x$
6.	$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x$ where $\lvert x \rvert < 1$

Example 6.1

Write the integration formula corresponding to

$$\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x.$$

Solution

The corresponding formula is

$$\int (-\operatorname{cosec}^2 x)\,dx = \cot x.$$

Remark

We have to be careful when considering functions whose domain is not the whole real line. For instance, when we say $\frac{d}{dx}(\log x) = \frac{1}{x}$, we know that $\log x$ is defined only for positive x, and, therefore, its derivative is also taken only for positive x. But, if we write $\int \frac{1}{x}\,dx = \log x$, the given function $\frac{1}{x}$ is defined for all non-zero values of x, and, therefore, its integral must also be defined there, but $\log x$ is not defined for negative values of x. This is overcome in the manner given below:

Formula 6.1

$$\int \frac{1}{x}\, dx = \log |x|$$

Note : The above formula is to be applied for x lying in an interval (δ, ∞) or an interval $(-\infty, -\delta)$ for a fixed $\delta > 0$. It does not apply for any $x \in \mathbf{R}$ or any x in an interval containing 0.

Proof

$$\frac{d}{dx}(\log x) = \frac{1}{x}, \quad x > 0$$

$$\frac{d}{dx}(\log(-x)) = \frac{1}{-x}(-1) = \frac{1}{x}, \quad x < 0$$

Combining these two, $$\frac{d}{dx}(\log |x|) = \frac{1}{x}, \quad x \neq 0$$

and, therefore, $$\int \frac{1}{x}\, dx = \log |x|$$

is one of the values of integral of $\frac{1}{x}$.

Remark

A function can have more than one integral. To see this consider

$$\frac{d}{dx}(e^x + 2) = e^x$$

and, therefore, $\int e^x\, dx = e^x + 2$. But already we have listed above that $\int e^x\, dx = e^x$. Thus, both e^x and $e^x + 2$ are integrals of the function e^x.

More generally, we can say that if

$$\int f(x)\, dx = g(x),$$

and if c is a constant, then it is also true that

$$\int f(x)\, dx = g(x) + c.$$

You can easily prove this.

Our next result is that apart from this there is no other function that is an integral of $f(x)$. In other words, if one integral of $f(x)$ is $g(x)$, then every integral on $f(x)$ must be of the form $g(x) \pm c$ for some constant c. To prove this, we require a theorem on differentiation, which we state and prove below.

Theorem 6.1

Let $f(x)$ be a function such that $f'(x)$ is identically zero. Then $f(x)$ is a constant function.

Proof

Let $a < b$ be two real numbers. Then, on the interval $[a, b]$, $f(x)$ satisfies the conditions of Lagrange's mean value theorem. Therefore, there is some c between a and b such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$. But by assumption, $f'(c) = 0$. Therefore, $f(b) = f(a)$. Thus, $f(b) = f(a)$ holds for any two real numbers a and b. This means that $f(x)$ is a constant function.

Corollary

Let $f(x)$ and $g(x)$ be two functions such that $f'(x) = g'(x)$ identically. Then there is a constant c such that $f(x) = g(x) + c$ identically, i.e. for all values of x.

Proof

$$\text{Let} \qquad h(x) = f(x) - g(x).$$

$$\text{Then} \qquad h'(x) = f'(x) - g'(x) = 0 \qquad \text{(by assumption)}$$

Therefore, by the above theorem, $h(x) =$ a constant c. Therefore,

$$f(x) = g(x) + h(x) = g(x) + c \text{ for all } x.$$

Corollary

Let $f(x)$ and $g(x)$ be two integrals of the same function. Then there is a constant c such that $f(x) = g(x) + c$ for all x.

Proof

This is only a restatement of the previous corollary.

Notation

In the sequel, we shall denote a constant by C without explicit mention.

$$\text{Thus} \qquad \int e^x \, dx = e^x + C.$$

By this, we mean: "For each real number C, $e^x + C$ is an integral of e^x and every integral of e^x is of this form". We say that the general integral of e^x is $e^x + C$.

Terminology

Symbols/terms/phrases	*Meaning*
$\int$	(indefinite) Integral
$\int f(x)\,dx$	Integral of $f(x)$ with respect to x
$f(x)$ in $\int f(x)\,dx$	Integrand
x in $\int f(x)\,dx$	Variable of integration
Integrate	Find the integrals
Integral of $f(x)$	A function $g(x)$ such that $g'(x) = f(x)$
Integration	The process of finding an integral
Arbitrary constant	Any real number C, considered as a constant function
Integral is unique upto a constant	Any two integrals of the same function differ only by a constant (see the last corollary)

Remark

Some rules of differentiation have their counterparts in integration.

Formula 6.2

$$\int [u(x) + v(x)]\,dx = \int u(x)\,dx + \int v(x)\,dx$$

Proof

We know
$$\frac{d}{dx}\,[f(x) + g(x)] = \frac{d}{dx}\,f(x) + \frac{d}{dx}\,g(x) \qquad \text{...(6.1)}$$

Let
$$u(x) = \frac{d}{dx}\,f(x) \text{ and } v(x) = \frac{d}{dx}\,g(x).$$

This gives
$$\int u(x)\,dx = f(x) \text{ and } \int v(x)\,dx = g(x).$$

Now,
$$\int u(x)\,dx + \int v(x)\,dx = f(x) + g(x)$$
$$= \int \frac{d}{dx}\,[f(x) + g(x)]\,dx$$

(by definition of integral)

$$= \int \left[\frac{d}{dx}\,f(x) + \frac{d}{dx}\,g(x)\right] dx \qquad \text{by (6.1)}$$

$$= \int [u(x) + v(x)]\,dx$$

Thus
$$\int [u(x) + v(x)]\,dx = \int u(x)\,dx + \int v(x)\,dx.$$

Remark

This is interpreted as follows: When u and v are two functions, their sum $u+v$ is defined pointwise by $(u+v)(x) = u(x) + v(x)$ for all x. Formula 6.2 can be stated in words as:
"an integral of the sum of two functions, is equal to the sum of integrals of these two functions"
For example,

$$\int (x^2 + \sin x)\, dx = \int x^2\, dx + \int \sin x\, dx.$$

Again, here and in the sequel, these equalities are only upto a constant. The right side of the equality may differ from the left side by a constant.

Formula 6.3

$$\int cg(x)\, dx = c \int g(x)\, dx, \text{ where } c \text{ is a real number.}$$

Proof

From the rule $\frac{d}{dx}\left(cf(x)\right) = c\,\frac{d}{dx} f(x)$, we obtain

$$\int cg(x)\, dx = c \int g(x)\, dx$$

(and as usual, the equality is upto a constant). For example,

$$\int 2x\, dx = 2\int x\, dx.$$

Formula 6.4

$$\int f'\left(g(x)\right) g'(x)\, dx = f\left(g(x)\right).$$

Proof

By the chain rule of differentiation, we have

$$\left(f\left(g(x)\right)\right)' = f'\left(g(x)\right) g'(x)$$

It, therefore, follows that

$$\int f'\left(g(x)\right) g'(x)\, dx = f\left(g(x)\right).$$

We shall see in Section 6.2 how this can be used to integrate some functions.

Note

Similarly by the product rule of differentiation, we have

$$\left(f(x)\, g(x)\right)' = f(x)\, g'(x) + g(x)\, f'(x).$$

It follows that

$$\int [f(x)\, g'(x) + g(x)\, f'(x)]\, dx = f(x)\, g(x).$$

We shall see in a later section how this can be rewritten so as to be useful for integrating the products of two functions.

Formula 6.5

$$\int x^n\, dx = \frac{x^{n+1}}{n+1} + C \text{ if } n \neq -1$$

Proof

$$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = \frac{1}{n+1}\,\frac{d}{dx}\,(x^{n+1})$$

$$= \frac{1}{n+1}\,(n+1)\,x^n$$

$$= x^n.$$

$$\therefore \text{ One of the values of } \int x^n\, dx \text{ is } \frac{x^{n+1}}{n+1}$$

$$\therefore \int x^n\, dx = \frac{x^{n+1}}{n+1} + C$$

[When n $= -1$, this proof fails because it involves $\frac{1}{n+1}$ See Formula 6.1 for this case.]

The following examples illustrate the previous results.

Example 6.2

Find $\int (2x^2 + e^x)\, dx$.

Solution

$$\int (2x^2 + e^x)\, dx = \int 2x^2\, dx + \int e^x\, dx$$

$$= 2\int x^2\, dx + \int e^x\, dx$$

$$= 2\left(\frac{x^3}{3}\right) + C_1 + (e^x + C_2)$$

$$= \frac{2x^3}{3} + e^x + C.$$

The two constants C_1 and C_2 are merged as a simple constant C.

Example 6.3

Find $\int \cos 2x\, dx$.

Solution

We look for a function whose derivative is $\cos 2x$. From our knowledge that the derivative of $\sin x$ is $\cos x$, we consider $\sin 2x$ first.

We know $\frac{d}{dx}(\sin 2x) = 2\cos 2x$

Therefore, $\cos 2x = \frac{1}{2}(2\cos 2x)$

$$= \frac{1}{2}\frac{d}{dx}(\sin 2x)$$

$$= \frac{d}{dx}(\frac{1}{2}\sin 2x).$$

Therefore, $\int \cos 2x\, dx = \frac{1}{2}\sin 2x + C_1.$

Example 6.4

Find $\int (\sin 2x - 4e^{3x})\, dx.$

Solution

$$\int (\sin 2x - 4e^{3x})\, dx = \int \sin 2x\, dx - \int 4e^{3x}\, dx.$$

Now, $\int \sin 2x\, dx = -\frac{1}{2}\cos 2x + C_1$

because $\frac{d}{dx}\left(-\frac{1}{2}\cos 2x\right) = \left(-\frac{1}{2}\right)(-\sin 2x)2.$

[To arrive at $-\frac{1}{2}\cos 2x$, we first find the derivative of $\cos 2x$, and then find out the constant $-\frac{1}{2}$ so as to make the derivative as $\sin 2x$]

Next,
$$\int 4e^{3x}\,dx = 4\int e^{3x}\,dx$$
$$= 4\left(e^{3x}\cdot\frac{1}{3}\right) + C_2.$$

[This is again, because we know
$$\frac{d}{dx}\left(e^{3x}\right) = 3e^{3x}$$
$$\therefore \frac{d}{dx}\left(\frac{e^{3x}}{3}\right) = e^{3x}]$$

Thus
$$\int (\sin 2x - 4e^{3x})\,dx = -\frac{1}{2}\cos 2x + \frac{4}{3}e^{3x} + C.$$

Comparison and Contrast between Differentiation and Integration

1. Both are operations on functions. Both give functions as answers.
2. Both are linear This means
   (a) In order to differentiate the sum of two functions, it is enough to differentiate the two functions separately, and then add the two derivatives obtained. Similarly, in order to integrate the sum of two functions, it is enough to integrate them separately and then add the two integrals obtained.

   (b) The constant can be taken outside the differential or integral sign as shown below:

$$\frac{d}{dx}(cf(x)) = c\,\frac{d}{dx}f(x)$$

and $$\int cf(x)\,dx = c\int f(x)\,dx.$$

3. You have already seen that not all functions are differentiable. Similarly, not all functions are integrable. You will learn about this in higher classes.
4. The derivative of a function, when it exists, is a unique function. The integral of a function is not so. It always varies upto a constant.
5. When a polynomial function $p(x)$ is differentiated, the result is a polynomial function whose degree is 1 less than that of $p(x)$. When a polynomial function $p(x)$ is integrated, the result is

a polynomial function whose degree is 1 more than that of $p(x)$.

6. Let $f(x)$ be a function obtainable in a finite number of steps, by taking sums, products, quotients, number-multiples and composites, from polynomial, exponential, logarithmic and trigonometric functions. For instance,

$$f(x) = \sin(\log x^2) + 2\log \operatorname{cosec} x^2.$$

We have already studied how to differentiate all such functions. However, it is not the case with integration. We do not have methods to integrate all such functions. We can integrate only some of them.

7. We can speak of the derivative of a function at a point. We never speak of the integral of a function at a point.

8. The derivative has a geometric meaning, namely, the slopes of the tangents to a curve. Similarly, the integral also has a geometric meaning, namely, the areas of some regions. This will be studied, in the next chapter.

9. The derivative is used for finding some physical quantities like the velocity of a moving particle. Similarly, the integral is also used in calculating some physical quantities like centre of mass, momentum, etc.

10. Differentiation is a process involving limits. So is integration, as will be seen in the next chapter.

11. Differentiation and integration are inverses of each other.

$$\frac{d}{dx}\left(\int f(x)\,dx\right) = f(x) \text{ and } \int\left(\frac{d}{dx}\,f(x)\right)dx = f(x) + C$$

whenever these exist. In other words, if for a function $f(x)$ both these processes are applied one after the other, we get back the function $f(x)$ upto a constant.

EXERCISE 6.1

1. For the following differentiation results, write down the corresponding integration results:

(a) $\frac{d}{dx}(\sec x) = \sec x \tan x$

(b) $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$

(c) $\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$

(d) $\frac{d}{dx}(\sin 2x) = 2\cos 2x$

(e) $\frac{d}{dx}\left(e^{x^2}\right) = 2x\, e^{x^2}$

2. For the following integration results write down the corresponding differentiation results:

(a) $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x$

(b) $\int e^{2x}\, dx = \frac{1}{2} e^{2x}$

(c) $\int \operatorname{cosec}(3x+4) \cot(3x+4)\, dx = -\frac{1}{3} \operatorname{cosec}(3x+4)$

(d) $\int 2 \sin x\, dx = -2 \cos x$

3. Write down an integral for each of the following:

(a) x^3

(b) x^{-5}

(c) $x^3 + x^{-5}$

(d) $4x^3 - 4x^{-5}$

(e) $\frac{1}{2x}$

(f) $x + \frac{1}{x}$

(g) $\frac{x^3 - 1}{x^2}$

(h) $ax^2 + bx + c$

4. Write down the general integral for each of the following:

(a) $\sec x \tan x$

(b) $\sin x + \cos x$

(c) $\sec x\, (\sec x + \tan x)$

(d) $x^{\frac{2}{3}} + 1$

5. Integrate the following, by trial and error, if necessary:
   - (a) $\sin 2x$
   - (b) e^{2x}
   - (c) $4e^{3x} + 1$
   - (d) $(ax + b)^2$

6.2 Integration by Substitution

In the last section, we integrated some functions like $\sin x$ and $\sin 2x$, by mere inspection. These functions were simple enough to make us guess their anti-derivatives. But more often, given a function $f(x)$, it may not be possible to find by inspection the function $F(x)$ whose derivative is $f(x)$ Hence, the need for developing methods of integration. In the rest of this chapter, some such methods are studied. Prominent among them are methods based on

1. Substitution (Sections 6.2 and 6.4)
2. Partial Fractions (Section 6.5)
3. Integration by parts (Section 6.6)
4. Successive Reduction (Section 6.8)

First, we consider the method of integration by substitution. This, again, is of two kinds. In both these kinds, when x is the variable of integration, we introduce a new variable y. In the first kind, the substitution is of the form $y = g(x)$. In the second kind, the substitution is of the form $x = g(y)$. These will be studied in Sections 6.2 and 6.4 respectively.

The first method of substitution is essentially the use of the formula 6.4, namely,

$$\int f'(g(x))\, g'(x)\, dx = f(g(x)) + C$$

obtained from the chain rule of differentiation. The following examples illustrate this method.

Example 6.5

Find $\int \sin x \cos x \, dx$.

Solution

Let $g(x) = \sin x$ (This is called the substitution)
Then

$$g'(x) = \cos x.$$

$$\therefore \quad \int \sin x \cos x \, dx = \int g(x)\, g'(x)\, dx$$

$$= \frac{(g(x))^2}{2} + C$$

$$= \frac{\sin^2 x}{2} + C$$

Check

$$\frac{d}{dx}\left(\frac{\sin^2 x}{2} + C\right) = \sin x \cos x$$

which is the integrand.

Remark

There are more convenient ways of working out this solution. One of them is as under :

Let $y = \sin x.$

Then $dy = \cos x \, dx.$

(This is a formal way of writing $\frac{dy}{dx} = \cos x$)

$$\therefore \int \sin x \cos x \, dx = \int y \, dy = \frac{y^2}{2} + C$$

$$= \frac{\sin^2 x}{2} + C$$

Another convenient way is as below :

$$\int \sin x \cos x \, dx = \int (\sin x) \, d(\sin x)$$

$$= \frac{(\sin x)^2}{2} + C.$$

Example 6.6

Find $\int \frac{e^x}{1 + e^{2x}} \, dx.$

Solution

Let $y = e^x$.
Then $dy = e^x \, dx.$

$$\therefore \int \frac{e^x}{1 + e^{2x}} \, dx = \int \frac{dy}{1 + y^2}$$

$$= \tan^{-1} y + C$$

$$= \tan^{-1} e^x + C.$$

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Remark

This solution can also be worked out more briefly as follows :

$$\int \frac{e^x}{1+e^{2x}}\ dx = \int \frac{1}{1+(e^x)^2}\, d(e^x)$$

$$= \tan^{-1}(e^x) + C.$$

Remark

It is often important to guess what will be a useful substitution. Usually we make a substitution for a function whose derivative also occurs in the integral. This is explained in the next example.

Example 6.7

Find $\int x^2 \sin x^3\, dx$.

Solution

If we substitute

$$y = \sin x^3,$$

then

$$dy = 3x^2 \cos x^3\, dx.$$

This is not available in the integrand. Therefore, we try a different substitution.

Let $y = x^3$.

Then $dy = 3x^2\, dx$ (Note that x^2 is occurring in the integrand).

$$\therefore \int x^2 \sin x^3\, dx = \int \sin x^3 \left(\frac{3x^2}{3}\right) dx$$

$$= \int \sin y \,.\, \frac{1}{3} \,.\, dy$$

$$= \frac{1}{3} \int \sin y\, dy$$

$$= \frac{1}{3}\,(-\cos y) + C$$

$$= -\frac{1}{3} \cos x^3 + C.$$

The Steps of the Substitution Method

1. Guess a suitable substitution $y = g(x)$. This results in the change of variable of integration and brings the integral to a standard form in the new variable.
2. Compute $g'(x)$ and $dy = g'(x)\,dx$.
3. Write the integrand as a function of y, to be integrated with respect to y.
4. Integrate it and get the answer as a function of y.
5. Substitute back $y = g(x)$ in the answer and write the answer in terms of x.

Remark

Sometimes we may have to make special (or tricky) substitutions as seen in the examples below :

Example 6.8

Find $\int \sec x\,dx$.

Solution

Let $y = \sec x + \tan x$.

Then
$$\frac{dy}{dx} = \sec x \tan x + \sec^2 x$$
$$= \sec x\,(\tan x + \sec x)$$
$$= y \sec x.$$

$$\therefore \quad \sec x\,dx = \frac{1}{y}\,dy.$$

$$\therefore \int \sec x\,dx = \int \frac{1}{y}\,dy$$

$$= \log |y| + C$$
$$= \log |\sec x + \tan x| + C.$$

Example 6.9

Integrate $\dfrac{1}{1 + \tan x}$

Solution

$$\begin{aligned}\frac{1}{1+\tan x} &= \frac{1}{1+\dfrac{\sin x}{\cos x}}\\ &= \frac{\cos x}{\cos x+\sin x}\\ &= \frac{(\cos x+\sin x)+(\cos x-\sin x)}{2(\cos x+\sin x)}\\ &= \frac{1}{2}+\frac{\cos x-\sin x}{2(\cos x+\sin x)}.\end{aligned}$$

Put $y = \cos x + \sin x.$

Then $dy = (\cos x - \sin x)\, dx.$

$$\begin{aligned}\therefore \int \frac{1}{1+\tan x}\,dx &= \int \frac{1}{2}\,dx + \int \frac{1}{2y}\,dy\\ &= \frac{x}{2}+\frac{1}{2}\log|y|+C\\ &= \frac{x}{2}+\frac{1}{2}\log|\cos x+\sin x|+C.\end{aligned}$$

Remark

In such problems, one has to guess the right substitution through practical experience. It is possible that there are many substitutions, and one may be more elegant than the other.

The following table gives the integrals of the six standard trigonometric functions. These will be used later without reference.

---

1. $\int \sin x\, dx = -\cos x + C$
2. $\int \cos x\, dx = \sin x + C$
3. $\int \tan x\, dx = -\log|\cos x| + C$
4. $\int \cot x\, dx = \log|\sin x| + C$
5. $\int \sec x\, dx = \log|\sec x + \tan x| + C$
6. $\int \operatorname{cosec} x\, dx = -\log|\operatorname{cosec} x + \cot x| + C$

EXERCISE 6.2

Integrate the following functions :

1. $\cot x$
2. $2x \sin (x^2 + 1)$
3. $\dfrac{\log x}{x}$
4. $\dfrac{\log (x^2)}{x}$
5. $\dfrac{(\log x)^2}{x}$
6. $\sin x \sin (\cos x)$
7. $\sin (ax + b) \cos (ax + b)$
8. $\sqrt{ax + b}$
9. $(4x + 2) \sqrt{x^2 + x + 1}$
10. $(3x - 4)^3$
11. $x \sqrt{x + 2}$
12. $\sec^2(7 - 4x)$
13. $\dfrac{\sin x}{1 + \cos x}$
14. $\dfrac{\sin x}{(1 + \cos x)^2}$
15. $\cot x \log \sin x$
16. $\dfrac{\sin (\tan^{-1} x)}{1 + x^2}$
17. $\tan x$
18. $\dfrac{2 + 3x}{3 - 2x}$
19. $\dfrac{e^{2x}}{e^{2x} - 2}$
20. $\operatorname{cosec} x$
21. $\dfrac{1}{1 + \cot x}$
22. $\dfrac{1}{1 - \tan x}$
23. $\dfrac{\tan x}{\sec x + \cos x}$
24. $\dfrac{2 \cos x - 3 \sin x}{6 \cos x + 4 \sin x}$

6.3 Integration Using Trigonometric Identities

When the integrand involves some trigonometric functions, we use some known identities to find the integrand, as illustrated through the following examples :

Example 6.10

Find $\int \sin^2 nx \, dx$.

Solution

We know $\cos 2\theta = 1 - 2\sin^2\theta$ and, therefore,

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$

$$\therefore \int \sin^2 nx\, dx = \int \frac{1-\cos 2nx}{2} dx$$

$$= \int \frac{1}{2}\, dx - \int \frac{1}{2} \cos 2nx\, dx$$

$$= \frac{1}{2}x - \frac{1}{4n}\sin 2nx + C.$$

Example 6.11

Find $\int \cos mx \cos nx\, dx$, where m and n are positive integers and $m \neq n$. What will happen if $m = n$?

Solution

We know the identity

$$\cos mx \cos nx = \frac{1}{2}[\cos(m+n)x + \cos(m-n)x].$$

$$\therefore \int \cos mx \cos nx\, dx = \frac{1}{2}\left[\int \cos(m+n)x\, dx + \int \cos(m-n)x\, dx\right]$$

$$= \frac{1}{2}\left[\frac{\sin(m+n)x}{m+n} + \frac{\sin(m-n)x}{m-n}\right]$$

(since $m \neq n$ as given)

When $m = n$, $\cos(m-n)x$ becomes 1 and, therefore, its integral is x.
Thus, when $m = n$

$$\int \cos mx \cos nx\, dx = \int \cos^2 mx\, dx$$

$$= \int \frac{1+\cos 2mx}{2} dx$$

$$= \frac{1}{2}\left[\frac{\sin 2mx}{2m} + x\right] + C.$$

Example 6.12

Integrate $\tan^2 x$.

Solution

$$\int \tan^2 x\,dx = \int (\sec^2 x - 1)\,dx$$

$$= \int \sec^2 x\,dx - \int 1\,dx$$

$$= \tan x - x + C.$$

EXERCISE 6.3

Integrate the following:

1. $\sin 3x \cos 4x$
2. $\sin^2(2x+5)$
3. $\cos 2x \cos 4x \cos 6x$
4. $\cos^3 x$
5. $\sin^3(2x+1)$
6. $\sin^3 x \cos^3 x$
7. $\sin x \sin 2x \sin 3x$
8. $\sin 4x \sin 8x$
9. $\dfrac{1-\cos x}{1+\cos x}$
10. $\dfrac{\cos x}{1+\cos x}$
11. $\sin^4 x$
12. $\cos^4 2x$
13. $\dfrac{\sin^2 x}{1+\cos x}$
14. $\sin mx \sin nx$, where m and n are distinct positive integers.
15. $\sin mx \cos nx$, where m and n are distinct positive integers.
16. $\sin^2 bx$
17. $\cot^2 x$
18. $(2\tan x - 3\cot x)^2$
19. $\dfrac{1}{\sin^2 x \cos^2 x}$
20. $\dfrac{1}{1+\sin x}$
21. $\dfrac{1}{1-\cos x}$

22. $\dfrac{\cos^9 x}{\sin x}$ 23. $\sin^5 x$

6.4 Some Special Integrals

In this section, we prove the following formulas and apply them for integrating many functions. In proving these formulas, no newer methods are required than what has been covered in the previous sections.

1. $\displaystyle\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C$
2. $\displaystyle\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + C$
3. $\displaystyle\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$
4. $\displaystyle\int \frac{dx}{\sqrt{x^2-a^2}} = \log\left|x+\sqrt{x^2-a^2}\right| + C$
5. $\displaystyle\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + C$
6. $\displaystyle\int \frac{dx}{\sqrt{x^2+a^2}} = \log\left|x+\sqrt{x^2+a^2}\right| + C$

Using these formulas, we show how to integrate functions of the form

$\dfrac{1}{ax^2+bx+c}$ and $\dfrac{1}{\sqrt{ax^2+bx+c}}$

Now let us prove the following formulas :

Formula 6.6

$$\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C.$$

Remark

The remark made under Formula 6.1 is relevant here and in Formula 6.7 below.

Proof

$$\frac{1}{x^2-a^2} = \frac{1}{(x-a)(x+a)}$$

$$= \frac{1}{2a}\left[\frac{(x+a)-(x-a)}{(x-a)(x+a)}\right]$$

$$= \frac{1}{2a}\left[\frac{1}{x-a} - \frac{1}{x+a}\right].$$

$$\therefore \int \frac{dx}{x^2-a^2} = \frac{1}{2a}\left[\int \frac{dx}{x-a} - \int \frac{dx}{x+a}\right]$$

$$= \frac{1}{2a}\Big[\log|x-a| - \log|x+a|\Big] + C$$

$$= \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C.$$

Formula 6.7

$$\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + C.$$

Proof

This can be deduced from formula 6.6 by observing

$$\int \frac{dx}{a^2-x^2} = -\int \frac{dx}{x^2-a^2}$$

and

$$\log\left|\frac{a+x}{a-x}\right| = -\log\left|\frac{x-a}{x+a}\right|$$

Formula 6.8

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C.$$

Proof

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a^2}\int \frac{dx}{1+\frac{x^2}{a^2}}.$$

Put $y = \frac{x}{a}$. Then $dy = \frac{1}{a}\ dx$.

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$$\therefore \quad \int \frac{dx}{a^2+x^2} = \frac{1}{a^2} \int \frac{a\,dy}{1+y^2} = \frac{1}{a} \tan^{-1} y + C$$

$$= \frac{1}{a} \tan^{-1} \frac{x}{a} + C.$$

Another Proof

Let $x = a \tan \theta.$

Then $dx = a \sec^2 \theta \, d\theta$

and $a^2 + x^2 = a^2 + a^2 \tan^2\theta$

$= a^2 (1 + \tan^2 \theta) = a^2 \sec^2 \theta$

$$\therefore \quad \int \frac{dx}{a^2+x^2} = \int \frac{a \sec^2 \theta \, d\theta}{a^2 \sec^2 \theta}$$

$$= \frac{1}{a} \int d\theta$$

$$= \frac{1}{a} \theta + C.$$

But our answer should not involve θ and it should be in terms of x. Since $x = a \tan \theta$, we have $\theta = \tan^{-1} \frac{x}{a}$. This gives

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a} \theta + C$$

$$= \frac{1}{a} \tan^{-1} \frac{x}{a} + C.$$

Formula 6.9

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{|a|} + C \quad (\text{taking } |x| < |a|).$$

Proof

Let $y = \dfrac{x}{|a|}.$

Then $dy = \dfrac{1}{|a|} dx$

and, therefore, $dx = |a| dy.$

Also $$\sqrt{a^2 - x^2} = \sqrt{a^2 \left(1 - \frac{x^2}{a^2}\right)}$$

$$= \sqrt{a^2 (1 - y^2)}$$

$$= |a| \sqrt{1 - y^2}$$

Thus $$\int \frac{dx}{\sqrt{a^2 - x}} = \int \frac{|a| \, dy}{|a| \sqrt{1 - y^2}} = \int \frac{dy}{\sqrt{1 - y^2}}$$

$$= \sin^{-1} y + C$$

$$= \sin^{-1} \frac{x}{|a|} + C.$$

Note that considering the cases where a is positive or negative we can write $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a} + C$ since $\sin^{-1}(-\alpha) = (2n + 1)\pi + \sin^{-1}(\alpha)$

Another Proof

Let $$x = a \sin \theta$$

Then $$dx = a \cos \theta \, d\theta$$

and $$\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2 \sin^2 \theta}$$

$$= \sqrt{a^2 \cos^2 \theta}$$

$$= a \cos\theta.$$

$$\therefore \int \frac{dx}{\sqrt{a^2 - x^2}} = \int \frac{a \cos \theta \, d\theta}{a \cos \theta}$$

$$= \int d\theta$$

$$= \theta + C$$

$$= \sin^{-1} \frac{x}{a} + C.$$

Formula 6.10

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \log |x + \sqrt{x^2 - a^2}| + C, \quad |x| > |a|$$

Proof

Let $x = |a| \sec \theta.$

Then $dx = |a| \sec \theta \tan \theta \, d\theta$

and $\sqrt{x^2 - a^2} = \sqrt{a^2 \sec^2 \theta - a^2}$

$$= \sqrt{a^2 \tan^2 \theta}$$

$$= |a| \tan \theta$$

$$\therefore \int \frac{dx}{\sqrt{x^2 - a^2}} = \int \frac{|a| \sec\theta}{|a \tan \theta|} \tan \theta \, d\theta$$

$$= \int \sec \theta \, d\theta$$

$$= \log |\sec \theta + \tan \theta| + C.$$

Note : The cancellation in the last but one step needs careful study of the behaviour of $\tan \theta$ in $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

Now our answer should not involve θ and must be in terms of x. Using $\tan \theta = \sqrt{\sec^2 \theta - 1}$, we write our answer as

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \log |\sec \theta + \tan \theta| + C$$

$$= \log \left| \frac{x}{|a|} + \sqrt{\frac{x^2}{a^2} - 1} \right| + C$$

$$= \log |x + \sqrt{x^2 - a^2}| + C.$$

Formula 6.11

$$\int \frac{dx}{\sqrt{x^2 + a^2}} = \log |x + \sqrt{x^2 + a^2}| + C.$$

Proof

This can be proved by the substitution $x = |a| \tan \theta$. This proof is left as an exercise.

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Method to Integrate Functions of the Form $\dfrac{1}{ax^2+bx+c}$

We first write

$$ax^2+bx+c=a\left[\left(x+\frac{b}{2a}\right)^2+\left(\frac{c}{a}-\frac{b^2}{4a^2}\right)\right]$$

and let $y=x+\dfrac{b}{2a}$

Then the integral reduces to the form $a\,(y^2+k^2)$. This can be integrated using the formulas just studied.

Example 6.13

Find $$\int\frac{dx}{1+x-x^2}.$$

Solution

$$\begin{aligned}1+x-x^2&=-(x^2-x-1)\\&=-\left[\left(x-\frac{1}{2}\right)^2-\frac{5}{4}\right]\\&=\frac{5}{4}-\left(x-\frac{1}{2}\right)^2\end{aligned}$$

$$\int\frac{dx}{1+x-x^2}=\int\frac{dx}{\left(\frac{\sqrt{5}}{2}\right)^2-\left(x-\frac{1}{2}\right)^2}$$

$$=\frac{1}{2\left(\frac{\sqrt{5}}{2}\right)}\log\frac{\frac{\sqrt{5}}{2}+x-\frac{1}{2}}{\frac{\sqrt{5}}{2}-x+\frac{1}{2}}+C$$

$$=\frac{1}{\sqrt{5}}\log\frac{\sqrt{5}-1+2x}{\sqrt{5}+1-2x}+C.$$

Example 6.14

Integrate $\frac{1}{1 + 2\,(x + 2)^2}$.

Solution

Let

$$y = \sqrt{2}\,(x + 2).$$
$$dy = \sqrt{2}\,dx.$$

$$\therefore \quad \int \frac{dx}{1 + 2\,(x + 2)^2} = \frac{1}{\sqrt{2}} \int \frac{dy}{1 + y^2}$$

$$= \frac{1}{\sqrt{2}}\,(\tan^{-1} y) + C$$

$$= \frac{1}{\sqrt{2}}\,\tan^{-1}\left(\sqrt{2}\,(x + 2)\right) + C.$$

Method to Integrate Functions of the Form $\frac{1}{\sqrt{ax^2 + bx + c}}$

This can be reduced to the form $\frac{1}{\sqrt{|\,a\,|}\,\sqrt{\pm\, y^2 \pm k^2}}$ and then integrated using the formulas just studied.

Example 6.15

Integrate $\frac{1}{\sqrt{a - bx^2}}$ (We assume a and b to be positive).

Solution

Let $\qquad y = \sqrt{b}\, x$

Then $\qquad dy = \sqrt{b}\, dx.$

$$\therefore \int \frac{dx}{\sqrt{a - bx^2}} = \int \frac{dy}{\sqrt{b}\,\sqrt{a - y^2}}$$

$$= \frac{1}{\sqrt{b}} \int \frac{dy}{\sqrt{(\sqrt{a})^2 - y^2}}$$

$$= \frac{1}{\sqrt{b}} \sin^{-1} \frac{y}{\sqrt{a}} + C$$

$$= \frac{1}{\sqrt{b}} \sin^{-1} \left(\frac{\sqrt{b}}{\sqrt{a}} x \right) + C.$$

Example 6.16

Find $\int \frac{dy}{\sqrt{2 - 4x + x^2}}$.

Solution

$$\int \frac{dy}{\sqrt{2 - 4x + x^2}} = \int \frac{dy}{\sqrt{(x-2)^2 - 2}}$$

$$= \log \left| (x-2) + \sqrt{(x-2)^2 - 2} \right| + C$$

$$= \log \left| (x-2) + \sqrt{x^2 - 4x + 2} \right| + C$$

In our next example, we use both the above formulas and the substitution method.

Example 6.17

Find $\int \frac{x^3 + x}{x^4 - 9}\, dx$.

Solution

$$\int \frac{x^3 + x}{x^4 - 9}\, dx = \int \frac{x^3}{x^4 - 9}\, dx + \int \frac{x}{x^4 - 9}\, dx$$

$$= I_1 + I_2$$

where, $I_1 = \int \frac{x^3}{x^4 - 9}\, dx$ and $I_2 = \int \frac{x}{x^4 - 9}\, dx$.

Let us first find I_1.

Let $y = x^4 - 9$.

Then $dy = 4x^3\, dx$.

$$\therefore \qquad I_1 = \int \frac{x^3}{x^4-9}\,dx = \frac{1}{4}\int \frac{dy}{y}$$

$$= \frac{1}{4}\log|y| + C_1$$

$$= \frac{1}{4}\log|x^4-9| + C_1.$$

Now let us find I_2.

Let $\quad t = x^2$

Then $\quad dt = 2x\,dx.$

$$\therefore \qquad I_2 = \int \frac{x\,dx}{x^4-9} = \frac{1}{2}\int \frac{dt}{t^2-9}$$

$$= \frac{1}{2}\left(\frac{1}{6}\log\left|\frac{t-3}{t+3}\right|\right) + C_2.$$

$$\therefore \int \frac{x^3+x}{x^4-9}\,dx = I_1 + I_2$$

$$= \frac{1}{4}\log|x^4-9| + \frac{1}{12}\log\left|\frac{x^2-3}{x^2+3}\right| + C.$$

Note : $C_1 + C_2$ is expressed as a single constant C.

Method to Integrate the Functions of the Form $\dfrac{px+q}{ax^2+bx+c}$ *or* $\dfrac{px+q}{\sqrt{ax^2+bx+c}}$

Express $px+q$ in the form $A(2ax+b)+B$ noting that $2ax+b$ is the derivative of ax^2+bx+c. Then the integral reduces to the known forms.

Example 6.18

Integrate $\dfrac{6x+7}{\sqrt{(x-5)(x-4)}}$

Solution

Let $\quad y = (x-5)(x-4) = x^2 - 9x + 20.$

Then $\quad dy = (2x-9)\,dx.$

The numerater $6x + 7$ can be written in the form

$$A(2x-9)+B$$

In fact, $6x + 7 = 3\,(2x - 9) + 34$ by easy calculation.

$$\therefore \quad \int \frac{6x+7}{\sqrt{(x-5)\,(x-4)}}\,dx = \int \frac{3\,dy}{\sqrt{y}} + \int \frac{34\,dx}{\sqrt{x^2-9x+20}}$$

$$= 3(2\sqrt{y}\,) + 34 \int \frac{dx}{\sqrt{\left(x-\frac{9}{2}\right)^2-\frac{1}{4}}}$$

$$= 6\sqrt{y} + 34 \log \left| x - \frac{9}{2} + \sqrt{x^2-9x+20} \right|$$

$$= 6\sqrt{x^2-9x+20} + 34 \log \left| x - \frac{9}{2} + \sqrt{x^2-9x+20} \right| + C.$$

Method to integrate $\dfrac{p(x)}{ax^2+bx+c}$ *, where p(x) is a polynomial of degree 2 or more*

First carry out the division and express it in the form

$$\text{Polynomial} + \frac{px+q}{ax^2+bx+c}$$

and then integrate.

Example 6.19

Integrate $\dfrac{x^2+x+1}{x^2-x}$.

Solution

$$\frac{x^2+x+1}{x^2-x} = \frac{x^2-x+2x+1}{x^2-x}$$

$$= 1 + \frac{2x+1}{x^2-x}.$$

Let $\qquad y = x^2 - x.$

Then $\qquad dy = (2x-1)\,dx.$

So, we write $2x + 1$ in the form $2x - 1 + 2$.

$$\therefore \int \frac{x^2+x+1}{x^2-x}\,dx = \int \left[1+\frac{2x-1}{x^2-x}+\frac{2}{x^2-x}\right] dx$$

$$= \int dx + \int \frac{dy}{y} + 2\int \frac{dx}{x^2-x}$$

$$= x + \log|y| + 2\int \frac{1}{\left(x-\frac{1}{2}\right)^2 - \frac{1}{4}}\,dx$$

$$= x + \log|x^2-x| + 2\,\frac{1}{2\left(\frac{1}{2}\right)}\log\left|\frac{x-\frac{1}{2}-\frac{1}{2}}{x-\frac{1}{2}+\frac{1}{2}}\right| + C$$

$$= x + \log|x^2-x| + 2\log\left|\frac{x-1}{x}\right| + C.$$

Summary of the Methods Given in this Section

S.No.	*Form of the integral*	*Method of reducing to the known forms*
1.	$\frac{1}{ax^2+bx+c}$	Letting $y = x + \frac{b}{2a}$, reduce the denominator to the form $a\,(y^2 \pm k^2)$
2.	$\frac{1}{\sqrt{ax^2+bx+c}}$	Letting $y = x + \frac{b}{2a}$, reduce the denominator to the form $\sqrt{\lvert a \rvert}\sqrt{\pm y^2 \pm k^2}$
3.	$\frac{px+q}{ax^2+bx+c}$	Express $px+q$ in the form $A(2ax+b)+B$
4.	$\frac{px+q}{\sqrt{ax^2+bx+c}}$	Express $px+q$ in the form $A\,(2ax+b)+B$
5.	$\frac{p(x)}{ax^2+bx+c}$ where degree of $p(x) \geq 2$	Carry out the division and express it as a polynomial + $\frac{px+q}{ax^2+bx+c}$

EXERCISE 6.4

Integrate the following functions :

1. $\dfrac{1}{\sqrt{4+x^2}}$

2. $\dfrac{1}{\sqrt{1+4x^2}}$

3. $\dfrac{1}{\sqrt{(2-x)^2+1}}$

4. $\dfrac{1}{\sqrt{(2-x)^2-1}}$

5. $\dfrac{1}{1+\dfrac{x^2}{4}}$

6. $\dfrac{1}{(x+2)^2+1}$

7. $\dfrac{1}{1-9x^2}$

8. $\dfrac{1}{9x^2-1}$

9. $\dfrac{1}{\sqrt{9-25x^2}}$

10. $\dfrac{1}{\sqrt{a^2+b^2x^2}}$

11. $\dfrac{1}{\sqrt{a^2-b^2x^2}}$

12. $\dfrac{1}{a^2-b^2x^2}$

13. $\dfrac{3x}{1+2x^4}$

14. $\dfrac{x^2}{1-x^6}$

15. $\dfrac{x^2}{\sqrt{x^6+4}}$

16. $\dfrac{x-1}{\sqrt{x^2+1}}$

17. $\dfrac{x-1}{\sqrt{x^2-1}}$

18. $\dfrac{ax^3+bx}{x^4+c^2}$

19. $\dfrac{x^2}{\sqrt{x^6-a^6}}$

20. $\dfrac{x^2}{\sqrt{x^6+a^6}}$

21. $\dfrac{1}{x^2+2x+7}$

22. $\dfrac{1}{4x^2-4x+3}$

23. $\dfrac{1}{9x^2+6x+1}$

24. $\dfrac{1}{x^2+4x+7}$

25. $\dfrac{1}{3x^2+13x-10}$

26. $\dfrac{1}{\sqrt{(x-1)(x-2)}}$

27. $\dfrac{1}{\sqrt{x(1-2x)}}$

28. $\dfrac{1}{\sqrt{2x-x^2}}$

29. $\dfrac{1}{\sqrt{8+3x-x^2}}$

30. $\dfrac{1}{\sqrt{(x-a)(x-b)}}$

31. $\dfrac{4x+1}{\sqrt{2x^2+x-3}}$

32. $\dfrac{x+2}{\sqrt{x^2-1}}$

33. $\dfrac{\cos x}{\sqrt{4-\sin^2 x}}$

34. $\dfrac{\sec^2 x}{\sqrt{16+\tan^2 x}}$

35. $\dfrac{x}{\sqrt{x^2+x+1}}$

36. $\dfrac{x^2}{\sqrt{x^2+1}}$

37. $\dfrac{x^2}{x^2+1}$

38. $\dfrac{x^2+5x+3}{x^2+3x+2}$

6.5 Integration through Partial Fractions

The functions given by algebraic expressions such as

$$\frac{4x^3-5}{(x-2)^3(x+2)}, \quad \frac{1}{(x+\sqrt{2})^3(x+3)} \text{ and } \frac{2x-5}{x^3-x^2+x-1}$$

are called rational functions. Here both the numerator and the denominator are polynomial functions. In this section, we give a method of integrating rational functions. For the sake of simplicity, we consider only those rational functions whose denominator has degree 2 or 3, or occasionally 4.

We already know how to integrate some simple rational functions such as

(i) polynomial functions (here the denominator is 1)

(ii) $\dfrac{A}{(x-a)^n}$, where n is a positive integer and A is a constant.

(iii) $\dfrac{px+q}{x^2+ax+b}$, where a, b, p, q are constants (See Section 6.4)

When the given integrand is a rational function (that may not be of the above simple types), it is always possible to write the integrand as a sum of simpler rational functions (like the ones listed above), by a method called *partial fraction decomposition*. After this, the integration can be carried out easily using the already known methods.

For this purpose, we presently explain the method of partial fraction decomposition through some examples.

The following table gives an idea what kind of partial fractions are to be taken for what kind of rational functions.

S.No.	*Form of the rational function*	*Form of the partial fraction*
1.	$\frac{px+q}{(x-a)\,(x-b)}$, $a \neq b$	$\frac{A}{x-a} + \frac{B}{x-b}$
2.	$\frac{px+q}{(x-a)^2}$	$\frac{A}{(x-a)} + \frac{B}{(x-a)^2}$
3.	$\frac{px+q}{x^2+ax+b}$, where x^2+ax+b cannot be factorised.	No reduction. Keep it as it is.
4.	$\frac{px^2+qx+r}{(x-a)\,(x-b)\,(x-c)}$ where a, b, c are distinct.	$\frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}$
5.	$\frac{px^2+qx+r}{(x-a)^2\,(x-b)}$, $a \neq b$	$\frac{A}{x-a} + \frac{B}{(x-a)^2} + \frac{C}{x-b}$
6.	$\frac{px^2+qx+r}{(x-a)^3}$	$\frac{A}{x-a} + \frac{B}{(x-a)^2} + \frac{C}{(x-a)^3}$
7.	$\frac{px^2+qx+r}{(x-a)(x^2+bx+c)}$, where x^2+bx+c cannot be factorised.	$\frac{A}{x-a} + \frac{Bx+C}{x^2+bx+c}$

Note that there are as many constants to be determined as the degree of the denominator is.

Example 6.20

Split $\frac{x}{(x+1)\,(x+2)}$ into partial fractions.

Solution

Let
$$\frac{x}{(x+1)\,(x+2)} = \frac{A}{x+1} + \frac{B}{x+2} \qquad \text{... (6.2)}$$

We shall now determine the constants A and B such that (6.2) holds. This can be done in many ways. We indicate three methods here.

Method 1

We rewrite the right side of (6.2) as

$$\frac{A(x+2)+B(x+1)}{(x+1)(x+2)}$$

and then equate the numerators of the two sides (noting that the denominators are already equal).

That is

$$A(x+2)+B(x+1)=x$$

$$x(A+B)+(2A+B)=x. \qquad \ldots (6.3)$$

Equating the coefficients of x and constant terms, we get

$$A+B=1$$

$$2A+B=0.$$

Solving these two equations we have $A=-1$ and $B=2$.

Method 2

We find the values of A and B satisfying (6.2) by giving specific values for x.

When $x=0$, it becomes $0=A+\frac{B}{2}$

and when $x=1$, it becomes $\frac{1}{6}=\frac{A}{2}+\frac{B}{3}$

Solving these two equations, we get $A=-1$ and $B=2$.

Method 3

$A=$ the value of $\frac{x}{x+2}$, when x is -1,

$$=\frac{-1}{-1+2}=-1$$

$B=$ the value of $\frac{x}{x+1}$, when x is -2,

$$=\frac{-2}{-2+1}=2$$

Explanation

The given rational function is $\frac{x}{(x+1)(x+2)}$. Omit the factor $x+1$. Put $x=-1$ (making $x+1=0$) in what remains, we get the value of A. Next, omit the factor $x+2$, put $x=-2$ (making $x+2=0$)

in what remains, we get the value of B.

Justification

In (6.3), if we put $x = -1$, we have

$$A\,(x+2) + B \cdot 0 = x.$$

$$\therefore \quad A = \frac{x}{x+2}\,, \text{ where } x \text{ is } -1.$$

Similarly
$$B = \frac{x}{x+1}\,, \text{ where } x \text{ is } -2.$$

Thus, substituting the values of A and B obtained above, we have

$$\frac{x}{(x+1)\,(x+2)} = \frac{-1}{x+1} + \frac{2}{x+2}\,.$$

Remark

The equation (6.2) above is an identity, that is a statement true for all (permissible) values of x. Some authors use the symbol $\equiv$ to indicate that the statement is an identity and use the symbol $=$ to indicate that the statement is an equation, i.e., to indicate that the statement is true only for certain values of x. In our discussion, we will use the symbol $=$ in both an identity and an equation. The context will tell in which sense the symbol is used.

Example 6.21

Split $\dfrac{x^2}{x^3+1}$ into partial fractions.

Solution

Here the denominator can be factorised as $(x+1)\,(x^2-x+1)$. It has a linear factor and a quadratic factor, not further reducible. We shall bring the given function to the form

$$\frac{x^2}{x^3+1} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1} \qquad \text{... (6.4)}$$

We rewrite the right side as

$$\frac{A(x^2-x+1) + (Bx+C)\,(x+1)}{(x+1)\,(x^2-x+1)}$$

and then equate the numerators as

$$x^2 = A(x^2-x+1) + (Bx+C\,)\,(x+1).$$

Comparing the corresponding coefficients, we get a system of three equations

$$A + B = 1$$

$$-A + B + C = 0$$

$$A + C = 0.$$

Solving this, we obtain $A = \frac{1}{3}$, $B = \frac{2}{3}$, $C = -\frac{1}{3}$. Thus, we have

$$\frac{x^2}{x^3+1} = \frac{1}{3(x+1)} + \frac{2x-1}{3(x^2-x+1)}$$

Remark

As in Example 6.20, here also there are two other methods to find the values of A, B and C.

Method 2

By putting $x = 0$ in (6.4), $0 = \frac{A}{1} + \frac{C}{1}$.

By putting $x = 1$ in (6.4), $\frac{1}{2} = \frac{A}{2} + \frac{B+C}{1}$.

By putting $x = 2$ in (6.4), $\frac{4}{9} = \frac{A}{3} + \frac{2B+C}{3}$.

These give the system of linear equations

$$A + C = 0$$

$$A + 2B + 2C = 1$$

$$A + 2B + C = \frac{4}{3}.$$

Solving this, we obtain $A = \frac{1}{3}$, $B = \frac{2}{3}$, $C = \frac{1}{3}$.

Method 3

$$A = \text{the value of } \frac{x^2}{x^2-x+1}, \text{ when } x \text{ is } -1,$$

$$= \frac{(-1)^2}{(-1)^2-(-1)+1} = \frac{1}{3}.$$

But there is no way to find B and C in this method. Therefore, we do not resort to this method in this problem.

Example 6.22

Write $\dfrac{x^2}{(x-1)^3(x+1)}$ in the form of partial fraction decomposition.

Solution

Here the denominator is completely factorised into linear factors, but some factors are repeated. Therefore, we take the form

$$\frac{x^2}{(x-1)^3(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} + \frac{D}{x+1}$$

and then proceed as in the previous examples. The right side can be written as

$$\frac{(x+1)\ [A\ (x-1)^2 + B\,(x-1) + C\,] + D(x-1)^3}{(x-1)^3\ (x+1)}$$

and so we get the equation

$$x^2 = [A\ (x-1)^2 + B\,(x-1) + C\,]\,(x+1) + D\,(x-1)^3.$$

When $x = 1$, this gives $1 = 2C$.

When $x = 0$, this gives $0 = A - B + C - D$.

When $x = 2$, this gives $4 = 3(A + B + C\,) + D$.

When $x = -1$, this gives $1 = -8D$.

Solving these four equations, we get

$$A = \frac{1}{8}, B = \frac{3}{4}, C = \frac{1}{2}\ , D = -\frac{1}{8}.$$

Therefore, we get

$$\frac{x^2}{(x-1)^3(x+1)} = \frac{1}{8(x-1)} + \frac{3}{4(x-1)^2} + \frac{1}{2(x-1)^3} - \frac{1}{8(x+1)}$$

Remark

There are many methods available for partial fraction decomposition, as seen in Example 6.20. In each problem, we use whatever method is convenient.

We can use the method of partial fraction decomposition to integrate rational fnctions. The following examples illustrate the procedure:

Example 6.23

Find $\displaystyle\int \frac{x^2dx}{(x-1)^3(x+1)}$

Solution

First we split the integrand into partial fractions. We have

$$\frac{x^2}{(x-1)^3(x+1)} = \frac{1}{8(x-1)} + \frac{3}{4(x-1)^2} + \frac{1}{2(x-1)^3} - \frac{1}{8(x+1)}$$

(See Example 6.22)

Therefore,

$$\int \frac{x^2\,dx}{(x-1)^3(x+1)} = \frac{1}{8}\int \frac{dx}{(x-1)} + \frac{3}{4}\int \frac{dx}{(x-1)^2} + \frac{1}{2}\int \frac{dx}{(x-1)^3} - \frac{1}{8}\int \frac{dx}{x+1}$$

$$= \frac{1}{8}\log|x-1| - \frac{3}{4(x-1)} + \frac{1}{2}\,\frac{-2}{(x-1)^2} - \frac{1}{8}\log|x+1| + C$$

$$= \frac{1}{8}\log\left|\frac{x-1}{x+1}\right| - \frac{3}{4(x-1)} - \frac{1}{(x-1)^2} + C.$$

Remark

In the rational expression $\frac{p(x)}{q(x)}$, if the degree of the numerator $p(x)$ is greater than or equal to the degree of the denominator $q(x)$, then we divide $p(x)$ by $q(x)$ and express

$$\frac{p(x)}{q(x)} = r(x) + \frac{p_1(x)}{q(x)},$$

where $r(x)$ is a polynomial function and $\frac{p_1(x)}{q(x)}$ is a rational function, degree of $p_1(x)$ being less than the degree of $q(x)$. Then we carry out integration after spliting $\frac{p_1(x)}{q(x)}$ into partial fractions. The following examples illustrate the procedure :

Example 6.24

Integrate $\frac{x^2+1}{x^2-5x+6}$

Solution

$$\frac{x^2+1}{x^2-5x+6} = 1 + \frac{5x-5}{x^2-5x+6}$$

$$= 1 + \frac{5x-5}{(x-2)(x-3)}.$$

Let

$$\frac{5x-5}{(x-2)(x-3)} = \frac{A}{x-3} + \frac{B}{x-2}.$$

$$A = \text{the value of } \frac{5x-5}{x-3}\text{, when } x \text{ is } 2\,,$$
$$= -5$$
$$B = \text{the value of } \frac{5x-5}{x-2}\text{, when } x \text{ is } 3,$$
$$= 10$$
$$\therefore \quad \frac{x^2+1}{x^2-5x+6} = 1 - \frac{5}{x-2} + \frac{10}{x-3}$$
$$\therefore \quad \int \frac{x^2+1}{x^2-5x+6}\,dx = 1\int dx - 5\int \frac{dx}{x-2} + 10\int \frac{dx}{x-3}$$
$$= x - 5\log|x-2| + 10\log|x-3| + C.$$

Now we consider an example, where a substitution will make the partial fraction decomposition easier.

Example 6.25

Integrate $\dfrac{x^2}{(x^2+1)(x^2+4)}$.

Solution

Let $y = x^2$.

Then
$$\frac{x^2}{(x^2+1)(x^2+4)} = \frac{y}{(y+1)(y+4)}$$
$$= \frac{A}{y+1} + \frac{B}{y+4}$$

where
$$A = \text{the value of } \frac{y}{y+4}\text{, when } y \text{ is } -1,$$
$$= -\frac{1}{3}$$

and
$$B = \text{the value of } \frac{y}{y+1}\text{, when } y \text{ is } -4,$$
$$= \frac{4}{3}$$

Thus
$$\frac{x^2}{(x^2+1)(x^2+4)} = \frac{-\frac{1}{3}}{y+1} + \frac{\frac{4}{3}}{y+4}$$
$$= -\frac{1}{3(x^2+1)} + \frac{4}{3(x^2+4)}$$
$$\therefore \quad \int \frac{x^2\,dx}{(x^2+1)(x^2+4)} = -\frac{1}{3}\int \frac{dx}{x^2+1} + \frac{4}{3}\int \frac{dx}{x^2+4}$$
$$= -\frac{1}{3}\tan^{-1}x + \frac{4}{3}\cdot\frac{1}{2}\tan^{-1}\frac{x}{2} + C.$$

$$= -\frac{1}{3}\tan^{-1}x + \frac{2}{3}\tan^{-1}\frac{x}{2} + C.$$

In the above example, the substitution was made only for the partial fraction part and not for the integration part. Now, we consider an example, where the integration involves a combination of the substitution method and the partial fraction method.

Example 6.26

Find $\displaystyle\int \frac{(3\sin\phi - 2)\cos\phi}{5 - \cos^2\phi - 4\sin\phi}\, d\phi.$

Solution

Let $\quad y = \sin\phi.$

Then $\quad dy = \cos\phi\, d\phi.$

Therefore,

$$\int \frac{(3\sin\phi - 2)\cos\phi}{5 - \cos^2\phi - 4\sin\phi}\, d\phi = \int \frac{(3y-2)\, dy}{5 - (1 - y^2) - 4y} \qquad (\cos^2\phi = 1 - \sin^2\phi)$$

$$= \int \frac{3y-2}{y^2 - 4y + 4}\, dy$$

$$= \int \frac{3y-2}{(y-2)^2}\, dy$$

$$= I \text{ (say)}$$

Now let $\quad \dfrac{3y-2}{(y-2)^2} = \dfrac{A}{y-2} + \dfrac{B}{(y-2)^2}.$

Then $\quad 3y - 2 = A(y-2) + B.$

Comparing the coefficients we get $A = 3$ and $B - 2A = -2.$

$\therefore \quad A = 3 \quad$ and $\quad B = 4.$

Therefore, the required integral is given by

$$I = \int \left[\frac{3}{y-2} + \frac{4}{(y-2)^2}\right] dy$$

$$= 3\int \frac{dy}{y-2} + 4\int \frac{dy}{(y-2)^2}$$

$$= 3\log|y-2| + 4\left(-\frac{1}{y-2}\right) + C$$

$$= 3\log|\sin\phi - 2| - \frac{4}{\sin\phi - 2} + C$$

$$= 3 \log |2 - \sin \phi| + \frac{4}{2 - \sin \phi} + C$$

(since $2 - \sin \phi$ is always positive)

EXERCISE 6.5

Integrate the following functions:

1. $\frac{1}{(x+1)(x+2)}$
2. $\frac{2x}{x^2 + 3x + 2}$
3. $\frac{x}{(x-1)(x-2)(x-3)}$
4. $\frac{x^2}{(x-1)(x-2)(x-3)}$
5. $\frac{1 - x^2}{x(1 - 2x)}$
6. $\frac{x^2 + x + 1}{(x-1)^3}$
7. $\frac{3x - 1}{(x-2)^2}$
8. $\frac{x^2 + x + 1}{x^2(x+2)}$
9. $\frac{x}{(x-1)^2(x+2)}$
10. $\frac{2}{(1-x)(1+x^2)}$
11. $\frac{3x - 2}{(x+1)^2(x+3)}$
12. $\frac{5x}{(x+1)(x^2-4)}$
13. $\frac{2x}{(x^3 - 1)}$
14. $\frac{x^3}{(x-1)(x-2)}$
15. $\frac{\cos x}{(1 - \sin x)(2 - \sin x)}$
16. $\frac{1}{x + x^2 + x^3}$
17. $\frac{1}{1 + x^4}$
18. $\frac{x^2 + 4}{x^4 + 16}$
19. $\frac{ax^2 + bx + c}{(x-a)(x-b)(x-c)}$
   a, b, c are distinct.
20. $\frac{bx + c}{(x-p)(x-q)(x-r)}$
   p, q, r are distinct

21. $\dfrac{2\sin 2\phi - \cos\phi}{6 - \cos^2\phi - 4\sin\phi}$

22. $\dfrac{\tan\phi + \tan^3\phi}{1 + \tan^3\phi}$

23. $\dfrac{x^3 + x + 1}{x^2 - 1}$

6.6 Integration by Parts

In this section, we describe one more method of integration, that is found to be quite useful in integrating products of functions.

The *main formula* (Formula 6.12) of this section is stated in any one of the following equivalent forms, and is called the formula of integration by parts.

Formula 6.12

$$\int u dv = uv - \int v\, du \qquad \dots (6.5)$$

or $$\int u(x)\, dv(x) = u(x)\, v(x) - \int v(x)\; du(x) \qquad \dots (6.6)$$

or $$\int u(x)\, v'(x)\, dx = u(x)\, v(x) - \int v(x)\, u'(x)\, dx \qquad \dots (6.7)$$

Proof

Consider the product $u(x)\, v(x)$. By the product rule of differentiation,

$$\frac{d}{dx}\big(u(x)\, v(x)\big) = u(x)\, v'(x) + v(x)\, u'(x).$$

Integrating throughout,

$$u(x)\, v(x) = \int u(x)\, v'(x)\, dx + \int v(x)\, u'(x)\, dx.$$

Therefore,

$$\int u(x)\, v'(x)\, dx = u(x)\, v(x) - \int v(x)\, u'(x)\, dx.$$

This proves the formula in the form (6.7). The forms (6.5) and (6.6) are saying the same thing with different notations.

Remark

When we want to use this formula for finding the integral of the product of two functions, we must remember that one of them is taken as $u(x)$ and the other should be expressed in the form $v'(x)$.

Therefore, it is useful to remember the formula in a slightly different form

$$\int u(x)\,v(x)\,dx = u(x)\int v(x)\,dx - \int\left[u'(x)\int v(x)\,dx\right]dx. \qquad \dots (6.8)$$

and more often in words as

Integral of the product of two functions
= First function × integral of the second function
– Integral of (the derivative of the first function × integral of the second function)

To prove (6.8) from (6.7),

let $$w(x) = v'(x).$$

Then $$v(x) = \int w(x)\,dx$$

With this substitution, the formula (6.7) becomes

$$\int u(x)w(x)\,dx = u(x)\int w(x)\,dx - \int\left[u'(x)\int w(x)\,dx\right]dx.$$

This is same as (6.8) except that the function $v(x)$ has been named as $w(x)$.

The advantage of the form (6.8) (when compared to (6.7) is that the given integrand is taken as the product $u(x)\,v(x)$, as it is.

Example 6.27

Find $\int x \sin x\,dx$.

Solution

Here first function = x
second function = $\sin x$.

Then, integral of the second function = $-\cos x$. Therefore, by the formula of integration by parts

$$\int x\sin x\,dx = x(-\cos x) - \int [1\cdot(-\cos x)]\,dx$$
$$= -x\cos x + \sin x + C.$$

Remark

Another way of writing the same solution is given below:

$$\int x\sin x\,dx = \int x\,d(-\cos x)$$

$$= x\,(-\cos x) - \int 1 \cdot (-\cos x)\,dx$$

(by the formula of integration by parts)

$$= -x \cos x + \sin x + C.$$

The difference in the proof is that here we used the formula in the form (6.7) instead of the form (6.8).

Note

The notation $dv\,(x)$ is consistent with the notation dx because when $v\,(x) = x$. $u\,(x)\,dv\,(x)$ is same as $u\,(x)\,dx$.

Even when the integrand is very simple, and is not a product the method of integration by parts is useful, as seen below:

Example 6.28

Find $\int \log x\,dx$.

Solution

To start with, we are unable to guess a function whose derivative is $\log x$.

We take first function $= \log x$

second function $= 1$.

Then the integral of the second function is x.

$$\therefore \int (\log x \cdot 1)\,dx = (\log x)\,(x) - \int \frac{1}{x} \cdot x\,dx$$

$$= x \log x - x + C.$$

Remark

When the integrand is a product of two functions any one of them may be taken as the first function. But only a proper choice will make the work simpler, in some cases, as seen below:

Example 6.29

Integrate xe^x.

Solution

Take

$$\text{first function} = x$$
$$\text{second function} = e^x.$$

$$\text{Integral of the second function} = e^x.$$

$$\therefore \quad \int xe^x\,dx = xe^x - \int 1 \cdot e^x\,dx$$
$$= xe^x - e^x + C.$$

Suppose we take on the contrary,

$$\text{first function} = e^x$$
$$\text{second function} = x.$$

$$\text{Integral of the second function} = \frac{x^2}{2}$$

$$\therefore \quad \int e^x x\,dx = e^x\,\frac{x^2}{2} - \int e^x\,\frac{x^2}{2}dx.$$

It is not easier to find $\int e^x\,\frac{x^2}{2}dx$. Therefore, we must choose x as our first function.

Example 6.30

Find $\dfrac{x\sin^{-1}x\,dx}{\sqrt{1-x^2}}$.

Solution

Take

$$\text{first function} = x$$
$$\text{second function} = \frac{\sin^{-1}x}{\sqrt{1-x^2}}$$

Then integral of the second function is obtained by letting

$$y = \sin^{-1}x$$

so that

$$dy = \frac{dx}{\sqrt{1-x^2}}.$$

$$\therefore \int \frac{\sin^{-1}x}{\sqrt{1-x^2}}\,dx = \int y\,dy = \frac{y^2}{2} = \frac{1}{2}\,(\sin^{-1}x)^2.$$

Now,

$$\int \frac{x \sin^{-1} x}{\sqrt{1-x^2}} dx = x \ \frac{1}{2} = (\sin^{-1} x)^2 - \int 1 \cdot \frac{1}{2} \ (\sin^{-1} x)^2 \, dx.$$

$$= \frac{x}{2} \ (\sin^{-1} x)^2 - \frac{1}{2} \int (\sin^{-1} x)^2 \, dx.$$

We are stuck up because we do not know how to integrate $(\sin^{-1} x)^2$ easily. Therefore, we try by changing the first and second function.

Let first function $= \sin^{-1} x$

and second function $= \dfrac{x}{\sqrt{1-x^2}}$.

Then integral of the second function is

$$\int \frac{x \, dx}{\sqrt{1-x^2}}.$$

Now put $y = 1 - x^2$.

Then $dy = -2x \, dx$.

Therefore,

$$\int \frac{x \, dx}{\sqrt{1-x^2}} = -\frac{1}{2} \int \frac{dy}{\sqrt{y}}$$

$$= -\sqrt{y}$$

$$= -\sqrt{1-x^2}$$

$$\therefore \int \frac{x \sin^{-1} x}{\sqrt{1-x^2}} dx = \sin^{-1} x \ (-\sqrt{1-x^2}) - \int \frac{1}{\sqrt{1-x^2}} (-\sqrt{1-x^2}) dx.$$

$$= -\sqrt{1-x^2} \sin^{-1} x + x + C$$

$$= x - \sqrt{1-x^2} \sin^{-1} x + C.$$

Remark

In the above example it is easier to make the substitution $\sin^{-1} x = y$ and then perform integration by parts. In the example given below, integration by parts is performed more than once and the required integral is calculated by solving an equation.

Example 6.31

Find $\int e^x \sin x \, dx$.

Solution

Let $$I = \int e^x \sin x \, dx.$$

Then $$I = \int \sin x \, d\,(e^x)$$

$$= \sin x \cdot e^x - \int e^x \cos x \, dx.$$

Let $$J = \int e^x \cos x \, dx.$$

Then $$I = \sin x \cdot e^x - J$$

and $$J = \int \cos x \, d\,(e^x)$$

$$= \cos x \cdot e^x - \int e^x (-\sin x) \, dx$$

$$= \cos x \cdot e^x + \int e^x \sin x \, dx$$

$$= \cos x \cdot e^x + I.$$

Thus we have
$$I = \sin x \cdot e^x - J \qquad \ldots (6.9)$$

and
$$J = \cos x \cdot e^x + I \qquad \ldots (6.10)$$

From (6.9) and (6.10).
$$I = \sin x \cdot e^x - \cos x \bullet e^x - I$$
$$\therefore \quad 2I = \sin x \cdot e^x - \cos x \bullet e^x$$
$$= e^x (\sin x - \cos x).$$

$$\therefore \quad I = \frac{1}{2} e^x (\sin x - \cos x) + C.$$

Remark

Simultaneously, we have calculated that
$$J = \cos x \cdot e^x + I$$

$$= \cos x \cdot e^x + \frac{1}{2} e^x (\sin x - \cos x) + C$$

$$= \frac{1}{2} e^x (\sin x + \cos x) + C.$$

EXERCISE 6.6

Integrate the functions 1 to 10.

1. $x \log x$
2. $x^2 \log x$
3. $x \log 2x$
4. $x \cos x$
5. $\sin^{-1} x$
6. $x \sin^{-1} x$
7. $x^2 \sin^{-1} x$
8. $x \tan^{-1} x$
9. $x \cos^{-1} x$
10. $(\sin^{-1} x)^2$
11. We know that x^3 is the integral of $3x^2$. Taking this as $(3x)(x)$, verify that the integration by parts gives the same answer.
12. Taking x^2 as $x^3 \left(\frac{1}{x}\right)$ integrate it by parts. Use this to find $\int x^2 \log x \, dx$.
13. Taking $\sin x$ as $\sin x \bullet 1$, integrate it by parts. Use this to find $\int x \cos x \, dx$.
14. Integrate $\sin x \cos x$ by parts. If the integral is I, you get $I = \sin^2 x - I$; from this find I. Check your answer by finding I by another method.
15. Find $\int e^{2x} \sin x \, dx$.
16. Find $\int e^{ax} \cos bx \, dx$.
17. Find $\int e^x (\sin x + \cos x) \, dx$.
18. Find $\int \sec^3 \theta \, d\theta$.

6.7 Integrals of Some More Types

In this section, we use known methods to find the following kinds of integrals:

Integral	*Substitution*
$\int \sqrt{a^2 - x^2} \, dx$	Substitute $x = a \sin \theta$

$\int \sqrt{a^2+x^2}\,dx$	Substitute $x = a\tan\theta$
$\int \sqrt{x^2-a^2}\,dx$	Substitute $x = a\sec\theta$
$\int \sqrt{ax^2+bx+c}\,dx$	Substitute $y = px + q$ for suitable p and q and bring it to one of the above three forms.
$\int (px+q)\sqrt{ax^2+bx+c}\,dx$	Write $px+q = A(2ax+b)+B$. Put $y = ax^2+bx+c$. Bring it to the form $A\int \sqrt{y}\,dy + B\int \sqrt{ax^2+bx+c}\,dx$
$\int \frac{px+q}{\sqrt{ax^2+bx+c}}\,dx$	Write $y = ax^2+bx+c$. Write $px+q = A(2ax+b)+B$. Bring it to the form $A\int \frac{dy}{\sqrt{y}} + B\int \frac{dx}{\sqrt{ax^2+bx+c}}$
$\int \frac{dx}{a+b\cos x}$	Put $t = \tan\frac{x}{2}$ Express $\cos x$ in terms of $\tan\frac{x}{2}$ $\left(\cos x = \frac{1-t^2}{1+t^2}\right)$
$\int \frac{dx}{a+b\sin x}$	Put $t = \tan\frac{x}{2}$ $\left(\sin x = \frac{2t}{1+t^2}\right)$

Example 6.32

Find $\int \sqrt{a^2-x^2}\,dx$, $|x| \le |a|$.

Solution

Let $$x = a\sin\theta$$

Then $$dx = a\cos\theta\, d\theta$$

and $$\sqrt{a^2-x^2} = \sqrt{a^2-a^2\sin^2\theta} = a\sqrt{1-\sin^2\theta} = a\cos\theta.$$

$$\therefore \int\sqrt{a^2-x^2}\,dx = \int a\cos\theta\cdot a\cos\theta\, d\theta$$

$$= a^2\int\cos^2\theta\, d\theta$$

$$= a^2\int\frac{1+\cos 2\theta}{2}\,d\theta$$

$$= \frac{a^2}{2}\left[\theta + \frac{\sin 2\theta}{2}\right] + C$$

$$= \frac{a^2}{2}\left[\theta + \sin\theta\cos\theta\right] + C$$

$$= \frac{a^2}{2}\left[\sin^{-1}\frac{x}{a} + \frac{x}{a}\sqrt{1-\frac{x^2}{a^2}}\right] + C$$

$$= \frac{a^2}{2}\sin^{-1}\frac{x}{a} + \frac{x}{2}\sqrt{a^2-x^2} + C.$$

Note: The procedure in the above example needs to be supplemented by an analysis as in the proof of Formula 6.10.

Example 6.33

Find $\int\sqrt{a^2+x^2}\,dx$

Solution

Put $$x = |a|\tan\theta,\ \frac{-\pi}{2} < \theta < \frac{\pi}{2}$$

Then $$dx = |a|\sec^2\theta\, d\theta$$

and $$\sqrt{a^2+x^2} = \sqrt{a^2+a^2\tan^2\theta} = |a\sec\theta|$$

Therefore,

$$\int \sqrt{a^2+x^2}\ dx = \int |a|\ |\sec\theta| \cdot |a| \sec^2\theta\, d\theta$$

$$= \int a^2\ |\sec^3\theta|\, d\theta \qquad \dots (6.11)$$

$$= a^2 \int \sec^3\theta\, d\theta.$$

To find $\int \sec^3\theta\, d\theta$ we resort to integration by parts taking $\sec\theta$ as the first function and $\sec^2\theta$ as the second.

$$\int \sec^3\theta\, d\theta = \sec\theta\tan\theta - \int \sec\theta\tan\theta\tan\theta\, d\theta$$

$$= \sec\theta\tan\theta - \int \sec\theta\ \tan^2\theta\, d\theta$$

$$= \sec\theta\tan\theta - \int \sec\theta\ (\sec^2\theta - 1)\, d\theta$$

$$= \sec\theta\ \tan\theta - \int \sec^3\theta\ d\theta + \int \sec\theta\, d\theta$$

$$= \sec\theta\ \tan\theta - \int \sec^3\theta\, d\theta + \log|\sec\theta + \tan\theta| + C_1.$$

Bringing all the terms involving integrals to the left,

$$2\int \sec^3\theta\, d\theta = \sec\theta\tan\theta + \log|\sec\theta + \tan\theta| + C_1.$$

Therefore,

$$\int \sec^3\theta\, d\theta = \frac{1}{2}\ [\sec\theta\tan\theta + \log|\sec\theta + \tan\theta|] + C$$

Taking this in (6.11) we have

$$\int \sqrt{a^2+x^2}\ dx = \frac{a^2}{2}\ [\sec\theta\tan\theta + \log|\sec\theta + \tan\theta|] + C$$

$$= \frac{a^2}{2}\left[\sqrt{1+\tan^2\theta}\tan\theta + \log\left|\tan\theta + \sqrt{1+\tan^2\theta}\right|\right] + C$$

$$= \frac{a^2}{2}\left[\frac{x}{a}\sqrt{1+\frac{x^2}{a^2}} + \log\left|\frac{x}{a} + \sqrt{1+\frac{x^2}{a^2}}\right|\right] + C$$

$$= \frac{1}{2}\left[x\sqrt{a^2+x^2} + a^2 \log\left|x + \sqrt{a^2+x^2}\right| + C_1\right]$$

Example 6.34

Find $\int \sqrt{x^2+2x+5}\,dx$.

Solution

$$\int \sqrt{x^2+2x+5}\;dx = \int \sqrt{(x+1)^2+4}\;dx$$

(This is of the form $\int \sqrt{a^2+y^2}\;dy$, where $y = x+1$ and $a = 2$)

$$= 2\log\left|x+1+\sqrt{x^2+2x+5}\right| + \frac{x+1}{2}\sqrt{x^2+2x+5} + C.$$

(as seen in Example 6.33)

Example 6.35

Find $\int \sqrt{x^2-a^2}\,dx,\ a > 0$

Solution

Let $x = a\sec\theta.$

Then $dx = a\sec\theta\tan\theta\,d\theta$

and $\sqrt{x^2-a^2} = \sqrt{a^2\sec^2\theta - a^2} = a\tan\theta.$

$$\therefore \int \sqrt{x^2-a^2}\;dx = \int a^2\sec\theta\tan^2\theta\,d\theta.$$

But

$$\int \sec\theta\tan^2\theta\,d\theta = \tan\theta\sec\theta - \int \sec^2\theta\;\sec\theta\,d\theta$$

(integrating by parts, taking $\tan\theta$ as the first function)

$$= \sec\theta\tan\theta - \int \sec^3\theta\,d\theta$$

$$= \frac{1}{2}\left[\sec\theta\tan\theta - \log\left|\sec\theta+\tan\theta\right|\right] + C_1$$

(as proved in Example 6.33)

$$\therefore \int \sqrt{x^2 - a^2}\, dx = \frac{a^2}{2}\ [\sec\theta \tan\theta - \log|\sec\theta + \tan\theta|] + C_1.$$

$$= \frac{a^2}{2}\left[\frac{x}{a}\sqrt{\frac{x^2}{a^2} - 1} - \log\left|\frac{x}{a} + \sqrt{\frac{x^2}{a^2} - 1}\right|\right] + C_1$$

$$= \frac{1}{2}x\sqrt{x^2 - a^2} - \frac{a^2}{2}\log|x + \sqrt{x^2 - a^2}| + C.$$

(*Note*: $\log\left|\dfrac{x + \sqrt{x^2 - a^2}}{a}\right|$ can be written as $\log|x + \sqrt{x^2 - a^2}| - \log|a|$ and then $-\log|a|$ may be merged with C_1 to give the constant C. The result can, in fact, be verified for $|x| > |a|$.

Example 6.36

Find $\int (x - 5)\sqrt{x^2 + x}\, dx.$

Solution

Let $\qquad y = x^2 + x.$

Then $\qquad dy = (2x + 1)\, dx.$

We write $x - 5$ in the form $A(2x + 1) + B$.

$$\therefore\ 2A = 1 \quad\text{and}\quad A + B = -5.$$

$$\text{Hence, } A = \frac{1}{2} \text{ and } B = \frac{-11}{2}.$$

Thus $\qquad x - 5 = \dfrac{1}{2}(2x + 1) - \dfrac{11}{2}.$

$$\therefore \int (x - 5)\sqrt{x^2 + x}\, dx \quad \int\left[\frac{1}{2}(2x + 1) - \frac{11}{2}\right]\sqrt{x^2 + x}\, dx$$

$$= \frac{1}{2}\int \sqrt{y}\ dy - \frac{11}{2}\int \sqrt{x^2 + x}\, dx$$

$$= \frac{1}{2}\,\frac{y^{\frac{3}{2}}}{\frac{3}{2}} - \frac{11}{2}\int \sqrt{x^2 + x}\, dx$$

$$= \frac{1}{3}y^{\frac{3}{2}} - \frac{11}{2}\int \sqrt{\left(x + \frac{1}{2}\right)^2 - \frac{1}{4}}\, dx$$

$$= \frac{1}{3}(x^2 + x)^{\frac{3}{2}}$$

$$- \frac{11}{2}\left[\frac{x + \frac{1}{2}}{2}\sqrt{x^2 + x} - \frac{1}{8}\log\left| x + \frac{1}{2} + \sqrt{x^2 + x}\right|\right] + C$$

(See Example 6.35)

Example 6.37

Integrate $\frac{2x}{\sqrt{(x+3)(x+1)}}$

Solution

Let $y = (x + 3)(x + 1) = x^2 + 4x + 3.$

Then $dy = (2x + 4)\,dx.$

Write the numerator $2x$ as $A(2x + 4) + B.$

Here $2x = (2x + 4) - 4.$

$$\therefore \int \frac{2x\,dx}{\sqrt{(x+3)(x+1)}} = \int \frac{dy}{\sqrt{y}} - 4\int \frac{dx}{\sqrt{x^2 + 4x + 3}}$$

$$= 2\sqrt{y} - 4\int \frac{dx}{\sqrt{(x+2)^2 - 1}}$$

$$= 2\sqrt{x^2 + 4x + 3} - 4\log\left|x + \sqrt{x^2 + 4x + 3}\right| + C.$$

Example 6.38

Find $\int \frac{dx}{a + b\cos x}$, where a and b are positive constants.

Solution

Let $t = \tan\frac{x}{2}.$

Then $dt = \frac{1}{2}\sec^2\frac{x}{2}\,dx = \frac{1}{2}(1 + t^2)\,dx.$

Also $\cos x = \dfrac{1-t^2}{1+t^2}$ (using a formula from trigonometry)

$$\therefore \int \frac{dx}{a+b\cos x} = \int \frac{2\,dt}{1+t^2}\cdot\frac{1}{a+b\dfrac{1-t^2}{1+t^2}}$$

$$= 2\int \frac{dt}{a(1+t^2)+b(1-t^2)}$$

$$= 2\int \frac{dt}{(a-b)t^2+a+b}$$

Consider the three cases:

Case 1

Let $a > b$. Then $a-b$ is positive. This integral is of the type $\int \dfrac{dy}{y^2+c^2}$ (because $a+b$ and $a-b$ are positive).

$$\therefore \text{The required integral} = \frac{2}{a-b}\int \frac{dt}{t^2+\dfrac{a+b}{a-b}}$$

$$= \frac{2}{a-b}\sqrt{\frac{a-b}{a+b}}\ \tan^{-1}\left(t\sqrt{\frac{a-b}{a+b}}\right)+C$$

$$= \frac{2}{\sqrt{a^2-b^2}}\ \tan^{-1}\left(\sqrt{\frac{a-b}{a+b}}\ \tan\frac{x}{2}\right)+C.$$

Case 2

Let $a < b$. Then the integral is of the form $\int \dfrac{dy}{c^2-y^2}$.

$$\text{The required integral} = \frac{2}{b-a}\int \frac{dt}{\dfrac{b+a}{b-a}-t^2}$$

$$= \frac{2}{b-a}\ \frac{1}{2\sqrt{\dfrac{b+a}{b-a}}}\ \log\frac{\sqrt{\dfrac{b+a}{b-a}}+t}{\sqrt{\dfrac{b+a}{b-a}}-t}+C$$

$$= \frac{1}{\sqrt{b^2 - a^2}} \log \frac{\sqrt{b + a} + t\sqrt{b - a}}{\sqrt{b + a} - t\sqrt{b - a}} + C$$

$$= \frac{1}{\sqrt{b^2 - a^2}} \log \frac{\sqrt{b + a} + \sqrt{b - a}\tan\frac{x}{2}}{(b + a) - (b - a)\tan\frac{x}{2}} + C.$$

Case 3

Let $a = b$. Then the required integral is

$$2\int \frac{dt}{a + x} = \frac{t}{a}$$

$$= \frac{1}{a}\tan\frac{x}{2}.$$

EXERCISE 6.7

Integrate the following functions:

1. $\sqrt{4 - x^2}$
2. $\sqrt{1 - 4x^2}$
3. $\sqrt{4 + x^2}$
4. $\sqrt{1 + \frac{x^2}{9}}$
5. $\sqrt{x^2 + 3x}$
6. $\sqrt{x^2 + 4x + 6}$
7. $\sqrt{x^2 + 4x + 1}$
8. $\sqrt{1 - 4x - x^2}$
9. $x\sqrt{x + x^2}$
10. $\frac{x}{\sqrt{(x + 1)(x - 2)}}$
11. $\frac{2x - 1}{\sqrt{(x + 1)(x - 2)}}$
12. $\frac{6x + 5}{\sqrt{6 + x - 2x^2}}$
13. $\frac{1}{1 + 2\cos\theta}$
14. $\frac{1}{2 + \cos\theta}$
15. $\frac{1}{1 - 2\sin\theta}$
16. $\frac{1}{4\cos\theta - 1}$

17. $\dfrac{1}{12 + 12\cos\theta}$

18. $\dfrac{1}{a + b\sin x}$, where a and b are positive.

6.8 Integration by Successive Reduction

Sometimes when we are integrating a function by parts, we get another integral similar in form to the one we started with. This can again be integrated by parts leading to yet another integral of the same form. By successively doing so, we may end up in an easily integrable function, finally leading to the result that we seek. In this section, we see, through examples, how this method is useful in finding the integrals of some important functions.

Reduction formula for $\int x^n e^{ax}\,dx$

For each positive integer m, let

$$I_m = \int x^m e^{ax}\,dx.$$

Now we want to find I_n. Integrating by parts, taking the first function as x^n and the second function as e^{ax}, we have

$$I_n = \int x^n e^{ax}\,dx = x^n \frac{e^{ax}}{a} - \int nx^{n-1}\frac{e^{ax}}{a}\,dx$$

$$= \frac{1}{a}x^n e^{ax} - \frac{n}{a} I_{n-1}$$

This gives a relation between I_n and I_{n-1}. This relation between I_n and I_{n-1} is called a *reduction formula*. From this reduction formula we get, by replacing n by $(n-1)$.

$$I_{n-1} = \frac{1}{a}x^{n-1}e^{ax} - \frac{n-1}{a} I_{n-2}$$

In the same way, we can get $I_{n-2}, I_{n-3}, \ldots$. Thus, we have the reduction formula (Formula 6.13)

Formula 6.13

$$\text{If } I_n = \int x^n e^{ax}\,dx \text{ then } I_n = \frac{1}{a}x^n e^{ax} - \frac{n}{a} I_{n-1}$$

Example 6.39

Using the reduction formula 6.13 find $\int x^3 e^{ax}\,dx$.

Solution

$$I_3 = \int x^3 e^{ax}\, dx$$

$$= \frac{1}{a} x^3 e^{ax} - \frac{3}{a} I_2$$

$$= \frac{1}{a} x^3 e^{ax} - \frac{3}{a}\left[\frac{1}{a} x^2 e^{ax} - \frac{2}{a} I_1\right]$$

$$= \frac{1}{a} x^3 e^{ax} - \frac{3}{a^2} x^2 e^{ax} + \frac{6}{a^2} I_1$$

$$= \frac{1}{a} x^3 e^{ax} - \frac{3}{a^2} x^2 e^{ax} + \frac{6}{a^2}\left[\frac{1}{a} xe^{ax} - \frac{1}{a} I_o\right]$$

$$= \frac{1}{a} x^3 e^{ax} - \frac{3}{a^2} x^2 e^{ax} + \frac{6}{a^3} xe^{ax} - \frac{6}{a^3} I_o$$

$$= \frac{1}{a} x^3 e^{ax} - \frac{3}{a^2} x^2 e^{ax} + \frac{6}{a^3} xe^{ax} - \frac{6}{a^4} e^{ax} + C$$

(since $I_o = \int e^{ax}\, dx = \frac{1}{a} e^{ax}$)

$$= \frac{e^{ax}}{a^4}(a^3 x^3 - 3a^2 x^2 + 6ax - 6) + C.$$

Reduction formula for $\int x^n \sin x\, dx$

Let for each positive integer m,

$$I_m = \int x^m \sin x\, dx.$$

Now,

$$I_n = \int x^n \sin x\, dx$$

$$= x^n(-\cos x) - \int nx^{n-1}(-\cos x)\, dx \qquad \text{(integrating by parts)}$$

$$= -x^n \cos x + n\int x^{n-1} \cos x\, dx$$

$$= -x^n \cos x + n\,[x^{n-1} \sin x - \int (n-1)\, x^{n-2} \sin x\, dx\,]$$

$$= -x^n \cos x + nx^{n-1} \sin x - n(n-1)\int x^{n-2} \sin x \, dx$$

$$= -x^n \cos x + nx^{n-1} \sin x - n(n-1)I_{n-2}.$$

Thus, we have the reduction formula (Formula 6.14).

Formula 6.14

$$\text{If } I_n = \int x^n \sin x \, dx, \text{ then}$$

$$I_n = -x^n \cos x + nx^{n-1} \sin x - n(n-1)I_{n-2}.$$

Example 6.40

Integrate $x^5 \sin x$, using the reduction formula 6.14 derived above.

Solution

Now, putting $n = 5$ in the reduction formula 6.14 we have

$$I_5 = -x^5 \cos x + 5x^4 \sin x - 20I_3,$$

where I_3 is given by the same formula as

$$I_3 = -x^3 \cos x + 3x^2 \sin x - 6I_1$$

where

$$I_1 = \int x \sin x \, dx$$

$$= x(-\cos x) - \int (-\cos x) \, dx$$

$$= -x \cos x + \sin x + C.$$

Thus

$$I_5 = -x^5 \cos x + 5x^4 \sin x$$

$$- 20[-x^3 \cos x + 3x^2 \sin x - 6(-x \cos x + \sin x)] + C$$

$$= \cos x \cdot (-x^5 + 20x^3 - 120x) + \sin x \cdot (5x^4 - 60x^2 + 120) + C.$$

Remark

You may wonder why the integral of $x^5 \sin x$ should have so many terms. You are advised to verify that the derivative of this long expression is actually $x^5 \sin x$.

Reduction formula for $I_{m,n} = \int \sin^m x \cos^n x \, dx$, *where m and n are natural numbers*

Let $u(x) = \cos^{n-1} x$

and $v(x) = \sin^m x \cos x.$

Then $$\int v(x)\,dx = \frac{\sin^{m+1} x}{m+1}$$

Therefore, integration by parts gives

$$I_{m,n} = \cos^{n-1}x \cdot \left(\frac{\sin^{m+1} x}{\text{m} + 1}\right) - \int (n-1)\cos^{n-2}x\,(-\sin x)\cdot \frac{\sin^{m+1} x}{m+1}\,dx$$

$$= \frac{1}{m+1}\left[\cos^{n-1}x \cdot \sin^{m+1} x + (n-1)\int \cos^{n-2}x \cdot \sin^{m+2} x\,dx\right]$$

$$= \frac{1}{m+1}\left[\cos^{n-1}x \cdot \sin^{m+1} x + (n-1)\int \cos^{n-2}x \cdot \sin^{m} x \cdot (1-\cos^2 x)dx\right]$$

$$= \frac{1}{m+1}\,[\cos^{n-1}x \cdot \sin^{m+1} x + (n-1)\,(I_{m,n-2} - I_{m,n})]$$

$$\therefore \quad I_{m,n} \times \frac{n-1}{m+1}\ I_{m,n} = \frac{1}{\text{m}+1}[\cos^{n-1}x \cdot \sin^{m+1} x + (n-1)\,I_{m,n-2}].$$

$$\therefore (m+n)I_{m,n} = \cos^{n-1}x \cdot \sin^{m+1} x + (n-1)I_{m,n-2}.$$

Thus, we have the reduction formula (Formula 6.15).

Formula 6.15

$$\text{If } I_{m,n} = \int \sin^m x \cos^n x\,dx, \text{ then}$$

$$(m+n)\,I_{m,n} = \cos^{n-1} x \cdot \sin^{m+1} x + (n-1)I_{m,n-2}.$$

Example 6.41

Find $\int \sin^5 x \cdot \cos^3 x\,dx$.

Solution

Let us first use the reduction formula 6.15 to integrate

$$I_{5,3} = \int \sin^5 x \cdot \cos^3 x\,dx..$$

$$\therefore\ 8I_{5,3} = \cos^2 x \cdot \sin^6 x + 2I_{5,1}$$

$$= \cos^2 x \cdot \sin^6 x + 2\int \sin^5 x \cdot \cos x \, dx.$$

$$= \cos^2 x \cdot \sin^6 x + 2\int \sin^5 x \cdot d\ (\sin x)$$

$$= \cos^2 x \cdot \sin^6 x + 2\ \frac{\sin^6 x}{6} + C_1$$

$$= \cos^2 x \cdot \sin^6 x + \frac{\sin^6 x}{3} + C_1$$

$$\therefore\ \ I_{5,3} = \frac{\cos^2 x \cdot \sin^6 x}{8} + \frac{\sin^6 x}{24} + C.$$

Remark

This example can be solved easily without the use of reduction formula, as shown below:

Put $y = \sin x$

Then $dy = \cos x \, dx$

$$\therefore \int \sin^5 x \cdot \cos^3 x \, dx = \int \sin^5 x \cdot (1 - \sin^2 x) \cdot \cos x \, dx$$

$$= \int y^5 (1 - y^2) \, dy$$

$$= \frac{y^6}{6} - \frac{y^8}{8} + C$$

$$= \frac{\sin^6 x}{6} - \frac{\sin^8 x}{8} + C.$$

In certain cases, substitutions like the one above work, but not in all cases. For example, to evaluate $\int \sin^4 x \cdot \cos^2 x \, dx$ try the substitution $y = \sin x$. It will fail. We use the reduction formula in such cases.

Example 6.42

Find $\int \sin^4 x \cdot \cos^2 x \, dx$.

Solution

Let $I_{4,2} = \int \sin^4 x \cdot \cos^2 x \, dx$. Then substituting for m and n in the reduction formula, we have

$$6I_{4,2} = \cos x \cdot \sin^5 x + I_{4,0}$$
$$= \cos x \cdot \sin^5 x + \int \sin^4 x \, dx.$$

Now, $\sin^4 x$ can be integrated in two ways.

First Method

$$\int \sin^4 x \, dx = \int \sin^3 x \cdot d(-\cos x)$$
$$= \sin^3 x \cdot (-\cos x) - \int (-\cos x) \cdot 3 \sin^2 x \cdot \cos x \, dx$$
$$= -\sin^3 x \cdot \cos x + 3 \int \sin^2 x \cdot \cos^2 x \, dx$$
$$= -\sin^3 x \cdot \cos x + \frac{3}{4} \int \sin^2 2x \, dx$$
$$= -\sin^3 x \cdot \cos x + \frac{3}{8} \int (1 - \cos 4x) dx$$
$$= -\sin^3 x \cdot \cos x + \frac{3}{8}\left(x - \frac{\sin 4x}{4}\right) + C$$
$$= -\sin^3 x \cdot \cos x + \frac{3x}{8} - \frac{3 \sin 4x}{32} + C.$$

Second Method

$$\int \sin^4 x \, dx = \int \left(\frac{1 - \cos 2x}{2}\right)^2 dx$$
$$= \int \frac{1}{4} [1 - 2\cos 2x + \cos^2 2x] \, dx$$
$$= \frac{1}{4} \int \left(1 - 2\cos 2x + \frac{1 + \cos 4x}{2}\right) dx$$
$$= \frac{1}{4} \int \left(\frac{3}{2} - 2\cos 2x + \frac{1}{2}\cos 4x\right) dx$$
$$= \frac{1}{4}\left(\frac{3x}{2} - \sin 2x + \frac{1}{8}\sin 4x\right) + C_1.$$

(It is an exercise in trigonometry to prove that these two answers are the same). Thus, we have

$$6I_{4,2} = \cos x \cdot \sin^5 x + \frac{1}{4}\left(\frac{3x}{2} - \sin 2x + \frac{1}{8}\sin 4x\right) + C_1$$

$$\therefore \quad I_{4,2} = \frac{1}{6}\cos x \cdot \sin^5 x + \frac{1}{24}\left(\frac{3x}{2} - \sin 2x + \frac{1}{8}\sin 4x\right) + C.$$

EXERCISE 6.8

Integrate the following functions (1 to 10):

1. $x^3 e^x$
2. $x^4 e^{2x}$
3. $x^2 e^{-x}$
4. $x^3 e^{-x}$
5. $x^3 \cos x$
6. $x^5 \cos x$
7. $x^3 \cos 2x$
8. $x^4 \cos 2x$
9. $(x-1)^2 e^x$
10. $(x-1)^2 e^{2x}$

11. Find a reduction formula for $\int \cos^n x\,dx$.

12. Find a reduction formula for $\int \sin^n x\,dx$.

13. Find $\int \sin^4 x \cos^4 x\,dx$.

14. Find $\int \sin^4 x \cos^3 x\,dx$.

15. Find $\int \sin^3 x \cos^4 x\,dx$.

16. Find $\int \sin^2 x \cos^6 x\,dx$.

17. Find $\int \sin^3 x \cos^{10} x\,dx$.

18. If $I_{m,n} = \int \sin^m x - \cos^n x\,dx$, prove the reduction formula

$$(m+n)I_{m,n} = -\sin^{m-1} x \cdot \cos^{n+1} x + (m-1)I_{m-2,n}$$

MISCELLANEOUS EXERCISE ON CHAPTER 6

Integrate the following functions:

1. $\sin^{-1} 2x$
2. $\tan^4 x$
3. $\tan^3 x$
4. $\cot^2 x$
5. $\cos^1 x$
6. $\dfrac{1}{3\cos x + 4\sin x}$
7. $\sqrt{e^x - 1}$
8. $\dfrac{x^2}{\sqrt{x-1}}$
9. $\dfrac{x^2}{(x-1)^{\frac{3}{2}}}$
10. $\dfrac{\sqrt{1+x^2}}{1-x^2}$
11. $\dfrac{\sin x}{\sin 4x}$
12. $\dfrac{1}{x^2\sqrt{1+x^2}}$
13. $\sqrt{\dfrac{a+x}{x}}$
14. $\dfrac{1}{(x-1)\sqrt{x^2+4}}$
15. $\dfrac{\cos x}{\sqrt{4-\sin^2 x}}$
16. $\dfrac{5x}{(x+1)(x^2+9)}$
17. $\dfrac{1}{x\log x}$
18. $\sin^5 x$
19. $\dfrac{\sin x}{\sin(x-\alpha)}$
20. $\sqrt{\cot x}$
21. $\dfrac{x^2+1}{(x+1)^2} \cdot e^x$
22. $\dfrac{2+\sin 2x}{1+\cos 2x} \cdot e^x$
23. $\dfrac{1}{x^4+x^2+1}$
24. $\dfrac{x^2+x+1}{(x+1)^2(x+2)}$
25. $\dfrac{1}{(x+1)\sqrt{x^2-1}}$
26. $\dfrac{1}{\sin x + \sin 2x}$
27. $\dfrac{1}{\sin x(3+2\cos x)}$
28. $\dfrac{x}{1+\sin x}$

29. $\dfrac{1}{1 + 3\sin^2 x}$

30. $e^{2x} \sin x$

31. $x \sin^{-1} x$

32. $\tan^{-1}\left(\dfrac{3x - x^3}{1 - 3x^2}\right)$

33. $\dfrac{1}{x^4 + 1}$

34. $\dfrac{1}{\sqrt{(x - \alpha)(x - \beta)}}$

35. $\dfrac{1}{3 + 2\sin x + \cos x}$

36. $\dfrac{\sin^{-1}\sqrt{x} - \cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x}}$

37. $\log(\log x) + \dfrac{1}{(\log x)^2}$

38. $\sqrt{\dfrac{1 - \sqrt{x}}{1 + \sqrt{x}}}$

39. $\dfrac{x^3 \sin(\tan^{-1} x^4)}{1 + x^8}$

40. $\dfrac{1}{x\,[6(\log x)^2 + 7\log x + 2]}$

41. $\dfrac{1}{\sqrt{\sin^3 x \sin(x + \alpha)}}$

42. $\left(\sqrt{\tan x} + \sqrt{\cot x}\right)$

43. Prove $\int \dfrac{3(x + 1)(x + \log x)^2}{x}\, dx = (x + \log x)^3$.

CHAPTER 7

Definite Integral

In the previous chapter we have defined the indefinite integral as an anti-derivative. We have studied various methods for finding this anti-derivative.

In this chapter we shall study what is called the definite integral of a function. We shall see that these two notions, namely anti-derivative (or indefinite integral) and definite integral of a function are closely related.

7.1 Definite Integral as a Limit of Sum

Let us consider the problem of finding the areas of bounded regions. We already know some formulas for finding the areas of regions bounded by triangles, rectangles, trapeziums, circles, etc. These are the regions bounded by very familiar curves like line segments, circles, etc.

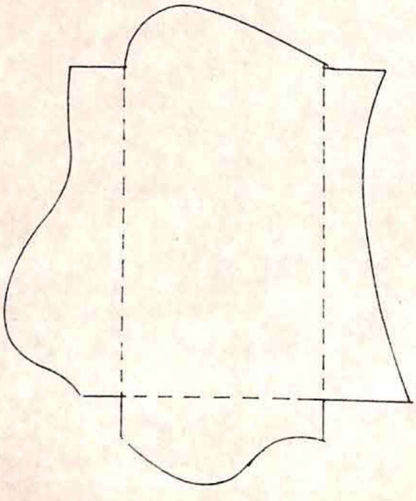

Fig. 7.1

We ask: Is there a way to find the areas of regions whose boundary is not familiar ? Suppose we know the equations of the boundary curves of the region (Fig. 7.1). Can we find the area of this region ? If so, how ?

As a first step in this direction, let us restrict our attention to finding the areas of such regions where the boundary which is not familiar to us is on only one side as in Fig. 7.2. This is because we expect that it is possible to divide any region into a few subregions of this kind, find the areas of these subregions and finally add up all these areas to get the area of the whole region (See Fig. 7.1).

Fig. 7.2

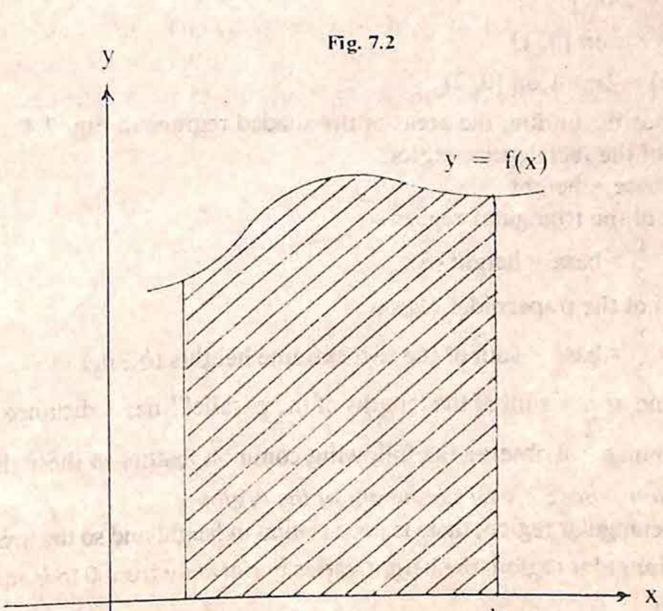

Fig. 7.3

Therefore, the problem of finding the area of a region may be posed as under:

Problem

Let $f(x)$ be a continuous function on a closed interval $[a, b]$. For the present, assume that all the values taken by the function are non-negative, so that the graph of the function is a curve above the x-axis (See Fig. 7.3). Consider the region between this curve, the x-axis and the ordinates $x = a$ and $x = b$, i.e. the shaded region in Fig. 7.3. Now the problem is to find the area of this shaded region.

In order to get an idea to solve this problem we consider three special cases of $f(x)$ (See Fig. 7.4).

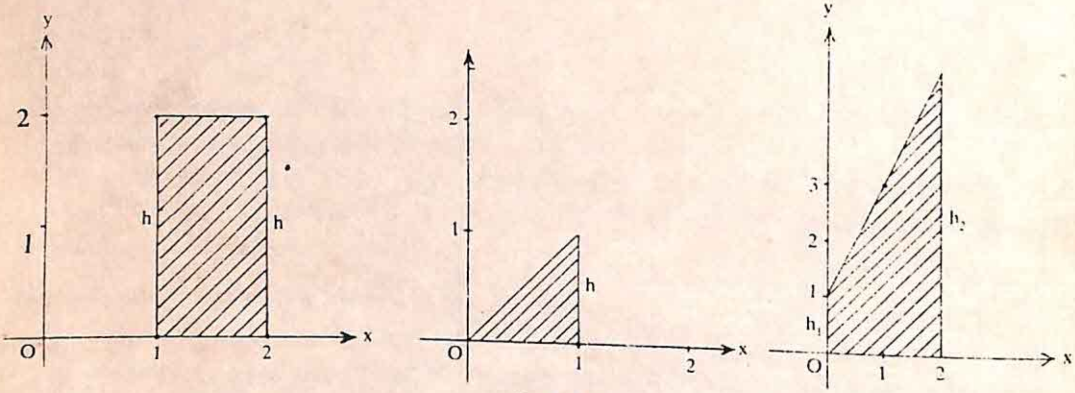

Fig. 7.4

In Fig. 7.4 (i), $f(x) = 2$ on $[1, 2]$.

In Fig. 7.4 (ii), $f(x) = x$ on $[0, 1]$.

In Fig. 7.4 (iii), $f(x) = 2x + 1$ on $[0, 2]$.

We know the formulas for finding the areas of the shaded regions in Fig. 7.4.

In Fig. 7.4 (i), area of the rectangular region

$$= \text{base} \times \text{height } (h).$$

In Fig. 7.4 (ii), area of the triangular region

$$= \frac{1}{2} \times \text{base} \times \text{height } (h).$$

In Fig. 7.4 (iii), area of the trapezoidal region

$$= \frac{1}{2} \times \text{base} \times \text{sum of the two extreme heights } (h_1, h_2).$$

[Note that this is same as $\frac{1}{2}$ × sum of the lengths of the parallel lines × distance between them.]

From these formulas, we observe the following common feature in these three cases.

Area of the region = base × average height of the region

In this case of rectangular region, there is no variation in height and so the average height is h.

In the case of triangular region, the height varies uniformaly from 0 to h and so the average height is $\frac{0 + h}{2}$ i.e. $\frac{h}{2}$.

In the case of trapezoidal region, the height varies uniformly from h_1 to h_2 and so the average height is $\frac{h_1+h_2}{2}$

From the three special cases of $f(x)$ considered above, we formulate the following rule for any function $f(x)$ on $[a, b]$:

Area of the bounded region (shaded region in Fig. 7.3)

= base × average height

The base is the length of the domain interval $[a, b]$.

The height at any point x is the value of $f(x)$ at that point.

Therefore, the average height is the average of the values taken by f in $[a, b]$. (This may not be so easy to find because the height may not vary uniformly.)

Our problem now is to find the average value of f in $[a, b]$.

Average Value of a Function in an Interval

If there are only finite number of values of f in $[a, b]$, we can easily get the average value by the following formula:

$$\text{Average value of } f \text{ in } [a, b] = \frac{\text{Sum of the values of } f \text{ in } [a, b]}{\text{Number of values}}.$$

But in our problem (Fig. 7.3), there are infinite number of values taken by f in $[a, b]$. How to find the average in such a case? The above formula does not help us, so we resort to estimate the average value of f in the following way:

First Estimate

Take the value of f at a only. The value is $f(a)$. We take this value, namely $f(a)$, as a rough estimate of the average value of f in $[a, b]$.

$$\text{Average value of } f \text{ in } [a, b] \text{ (first estimate)} = f(a) \qquad \ldots(7.1)$$

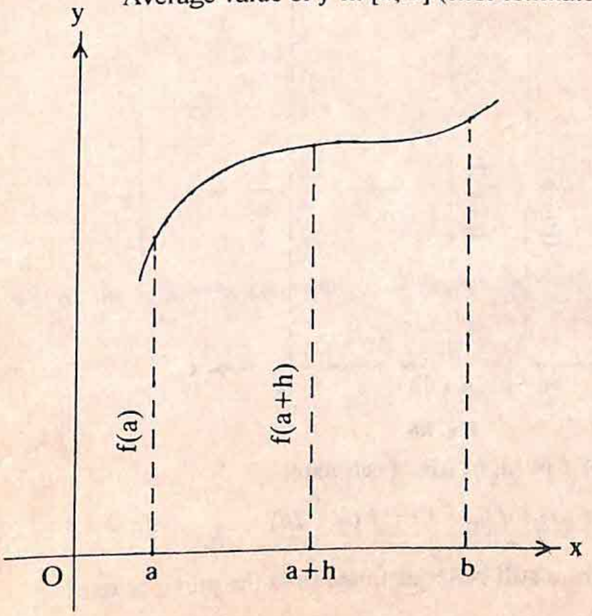

Fig. 7.5

Second Estimate

Divide $[a, b]$ into two equal parts or subintervals. Let the length of each subinterval be h.

So $$h = \frac{b-a}{2}.$$

Take the values of f at the left and points of the two sub-intervals. The values are $f(a)$ and $f(a+h)$ (Fig. 7.5). Take the average of these two values as the average value of f in $[a, b]$.

Average value of f in $[a, b]$ (second estimate)

$$= \frac{f(a) + f(a+h)}{2}, \quad h = \frac{b-a}{2} \qquad \dots (7.2)$$

This second estimate is expected to be a better estimate than the first.

Third Estimate

Divide $[a, b]$ into three equal sub-intervals of length h (Fig. 7.6). So, $h = \frac{b-a}{3}$. Take the values of f at the left end points of these three sub-intervals. The values are $f(a)$, $f(a+h)$, $f(a+2h)$. Take the average of these three three values as the average of f in $[a, b]$.

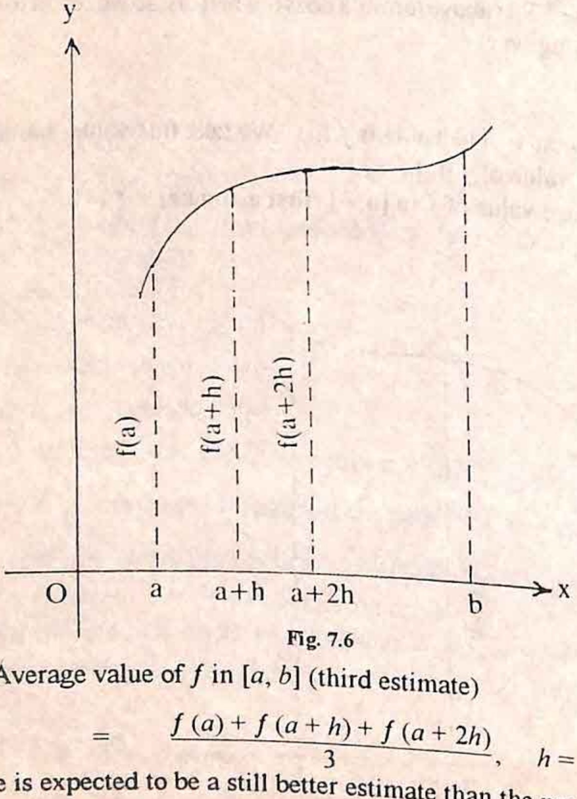

Fig. 7.6

Average value of f in $[a, b]$ (third estimate)

$$= \frac{f(a) + f(a+h) + f(a+2h)}{3}, \quad h = \frac{b-a}{3} \qquad \dots (7.3)$$

This third estimate is expected to be a still better estimate than the previous ones.

nth Estimate

Let n be any natural number. Divide the interval $[a, b]$ into n equal sub-intervals of length h (Fig. 7.7). So, $h = \frac{b-a}{n}$. Take the values of f at the left end points of the n sub-intervals. The values are $f(a), f(a+h), \ldots, f(a+\overline{n-1}\ h)$. Take the average of these n values as the average values of f $[a, b]$.

Average value of f in $[a, b]$ (nth estimate)

$$= \frac{f(a) + f(a+h) + f(a+2h) + \ldots + f(a+\overline{n-1}h)}{n}, \; h = \frac{b-a}{n} \qquad \ldots (7.4)$$

For larger values of n, (7.4) is expected to be a better estimate of what we seek as the average value of f in $[a, b]$.

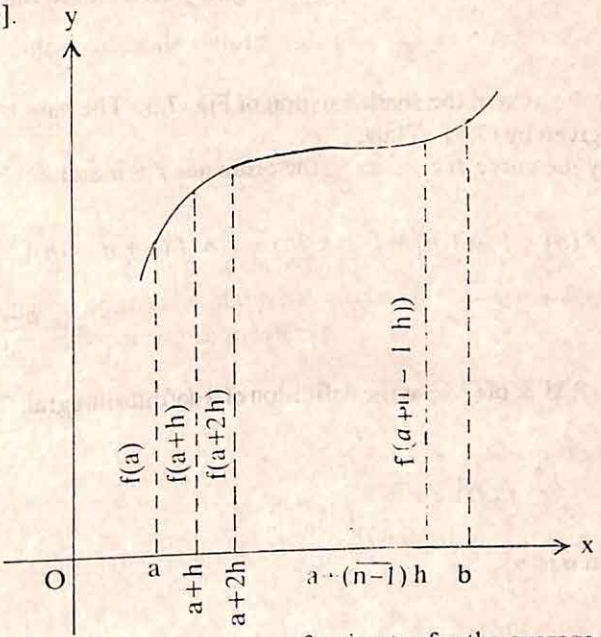

Fig. 7.7

Thus, we get the following sequence of estimates for the average value of f in $[a, b]$:

$$f(a)$$

$$\frac{1}{2}\,[f(a) + f(a+h)], \qquad h = \frac{b-a}{2}$$

$$\frac{1}{3}\,[f(a) + f(a+h) + f(a+2h)], \qquad h = \frac{b-a}{3}$$

……………………

……………………

$$\frac{1}{n}\ [f(a)+f(a+h)+\ldots+f(a+\overline{n-1}h)], \qquad h=\frac{b-a}{n}.$$

As we go farther and farther along this sequence, we are going closer and closer to our destination, namely, the average value taken by f in $[a, b]$. Therefore, it is reasonable to take that the limit of these estimates is the average value taken by f in $[a, b]$. In other words:

Average value of f in $[a, b]$

$$=\lim_{n\to\infty}\frac{1}{n}\ [f(a)+f(a+h)+f(a+2h)+\ldots+f(a+\overline{n-1}h)], \qquad h=\frac{b-a}{n} \quad \ldots (7.5)$$

It can be proved that this limit exists for all continuous functions f on a closed interval $[a, b]$.

Definite Integral

Now, we have the formula for the area of the shaded region of Fig. 7.3. The base is $(b-a)$ and the average height (value) is given by (7.5). Thus,
Area of the region bounded by the curve $f(x)$, x-axis, the ordinates $x=a$ and $x=b$

$$=(b-a)\cdot\lim_{n\to\infty}\frac{1}{n}[f(a)+f(a+h)+f(a+2h)+\ldots+f(a+\overline{n-1}h)],$$

$$h=\frac{b-a}{n} \quad \ldots (7.6)$$

We take the expression on R.H.S. of (7.6) as the definition of a definite integral. This definite integral is denoted by

$$\int_a^b f(x)\,dx$$

read as: 'integral of $f(x)$ from a to b'.

Definition

$$\int_a^b f(x)\,dx=(b-a)\ \lim_{n\to\infty}\frac{1}{n}\ [f(a)+f(a+h)+\ldots+f(a+\overline{n-1}h)],$$

$$h=\frac{b-a}{n} \quad \ldots (7.7)$$

Why do we bring in the term 'integral' here ? There is a reason. This is closely related to the integral that we studied in Chapter 6. This relation is not obvious now. It will be discovered in the next section.

Remark

In obtaining the estimates of the average values of f in $[a, b]$, we have taken the left end points of the sub-intervals. Why left end points ? Why not right end points of the sub-intervals ? We can as well take the right end points of the subintervals throughout and we get

$$\int_a^b f(x)\,dx = \lim_{n \to \infty} h\,[f(a+h) + f(a+2h) + \dots + f(b)], \qquad h = \frac{b-a}{n} \quad \dots (7.8)$$

It is true that (7.7) and (7.8) give the same limit. The proof is beyond the scope of this book.

Definite Integral through Areas of Rectangles

The above definition of the definite integral (7.7) can be explained in another way also. We rewrite (7.7) as

$$\int_a^b f(x)\,dx = \lim_{n \to \infty} [hf(a) + hf(a+h) + \dots + hf(a+\overline{n-1}h)], \qquad h = \frac{b-a}{n} \quad \dots (7.9)$$

Here the first term is $hf(a)$. It is the area of the rectangle marked as 1 in Figure 7.8 (because h and $f(a)$ are the adjacent sides of this rectangle). Similarly the second term $hf(a+h)$ is the area of the rectangle marked as 2 in the figure.

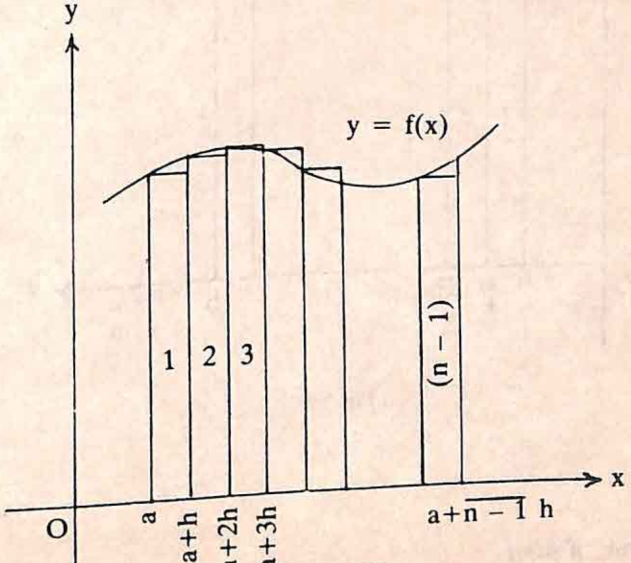

Fig. 7.8

Thus, $hf(a) + hf(a+h) + \ldots + hf(a + \overline{n-1}h)$ is the sum of the areas of these $n - 1$ rectangles marked in Fig. 7.8. The union of these rectangles is approximately the region between the curve and the x-axis. When n is larger, the number of rectangles is more, and the approximation is closer.

Therefore, when we take the limit as $n \to \infty$, we obtain that $\int_a^b f(x)\,dx$ as in (7.7) is the area of the region bounded by the four curves

$$\begin{aligned} & y = f(x) \\ & y = 0 \\ & x = 0 \\ \text{and} \quad & x = b. \end{aligned}$$

What will happen if we take the right end points instead of the left ? Then also, we get the same area as the limit of areas of unions of some other rectangles (See Fig. 7.9). This explains that $\int_a^b f(x)\,dx$ as in (7.8) is the area of the same region.

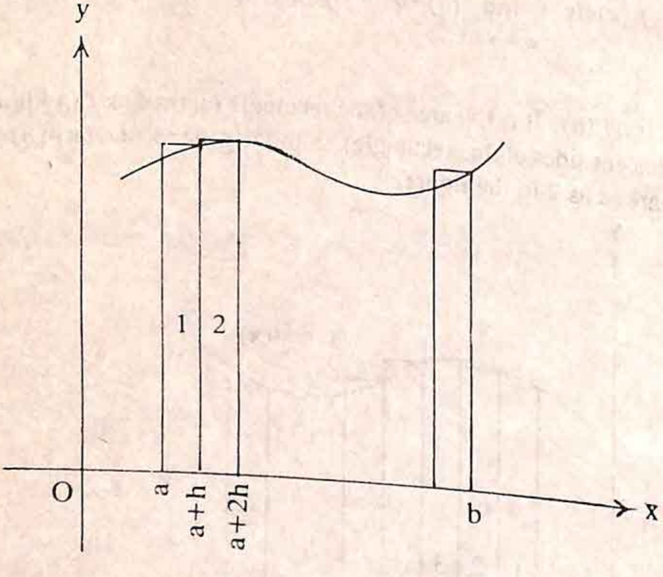

Fig. 7.9

Example 7.1

Find $\int_1^2 x\,dx$ as the limit of sums.

Solution

By definition,

$$\int_a^b f(x)\,dx = (b-a)\lim_{n\to\infty}\frac{1}{n}[\,f(a)+f(a+h)+\dots+f(a+\overline{n-1}h)\,],$$

$$h=\frac{b-a}{n}$$

In this example $a=1$, $b=2$, $f(x)=x$ and $h=\dfrac{b-a}{n}=\dfrac{1}{n}$.

$$\therefore\quad \int_1^2 x\,dx = \lim_{n\to\infty}\frac{1}{n}\left[f(1)+f\left(1+\frac{1}{n}\right)+\dots+f\left(1+\frac{n-1}{n}\right)\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\left[1+\left(1+\frac{1}{n}\right)+\left(1+\frac{2}{n}\right)+\dots+\left(1+\frac{n-1}{n}\right)\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\left[\underbrace{(1+1+\dots+1)}_{n\text{ times}}+\left(\frac{1}{n}+\frac{2}{n}+\dots+\frac{n-1}{n}\right)\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\left[n+\frac{1}{n}(1+2+\dots+\overline{n-1})\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\left[n+\frac{1}{n}\cdot\frac{(n-1)(n)}{2}\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\left[\frac{3n-1}{2}\right]$$

$$=\lim_{n\to\infty}\left[\frac{3}{2}-\frac{1}{2n}\right]$$

$$=\frac{3}{2}.$$

Remark

As you can see, $\int_1^2 x\,dx$ is the area of the shaded trapezoidal region in Fig. 7.10. We can find this area by using the formula also. We get the area as $\frac{1}{2}(2-1)(1+2)$ i.e. $\frac{3}{2}$. Thus, the answer we got is verified.

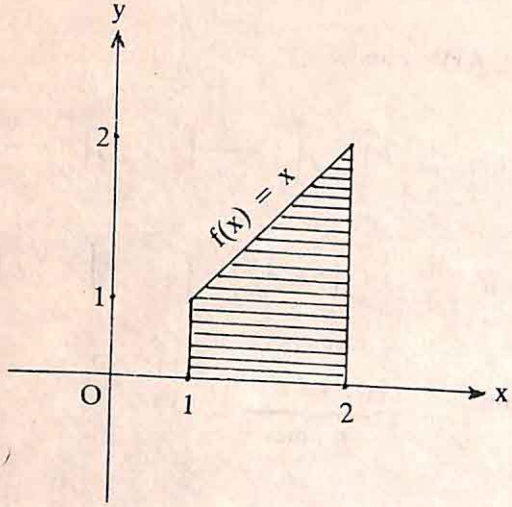

Fig. 7.10

Example 7.2

Find $\int_0^2 (x^2+1)\,dx$ as the limit of sums.

Solution

By definition

$$\int_a^b f(x)\,dx = (b-a)\lim_{n\to\infty}\frac{1}{n}\left[f(a)+f(a+h)+\ldots+f(a+\overline{n-1}h)\right],$$

$$h = \frac{b-a}{n}$$

In this example, $a = 0$, $b = 2$, $f(x) = x^2 + 1$, $h = \frac{2-0}{n} = \frac{2}{n}$.

$$\int_0^2 (x^2+1)\,dx = 2\lim_{n\to\infty}\frac{1}{n}\left[f(0)+f\left(\frac{2}{n}\right)+f\left(\frac{4}{n}\right)+\ldots+f\left(\frac{2n-2}{n}\right)\right]$$

$$= 2 \lim_{n \to \infty} \frac{1}{n} \left[1 + \left(\frac{2^2}{n^2} + 1 \right) + \left(\frac{4^2}{n^2} + 1 \right) + \ldots + \left(\frac{(2n-2)^2}{n^2} + 1 \right) \right]$$

$$= 2 \lim_{n \to \infty} \frac{1}{n} \left[\underbrace{(1 + 1 + \ldots +)}_{n \text{ times}} + \frac{1}{n^2} \left(2^2 + 4^2 + \ldots + (2n-2)^2\right) \right]$$

$$= 2 \lim_{n \to \infty} \frac{1}{n} \left[n + \frac{2^2}{n^2} \left(1^2 + 2^2 + \ldots + (n-1)^2\right) \right]$$

$$= 2 \lim_{n \to \infty} \frac{1}{n} \left[n + \frac{4}{n^2} \cdot \frac{(n-1)\, n\, (2n-1)}{6} \right]$$

$$= 2 \lim_{n \to \infty} \frac{1}{n} \left[n + \frac{2}{3} \frac{(n-1)(2n-1)}{n} \right]$$

$$= 2 \lim_{n \to \infty} \left[1 + \frac{2}{3} \frac{(n-1)(2n-1)}{n^2} \right]$$

$$= 2 \lim_{n \to \infty} \left[1 + \frac{2}{3} \left(1 - \frac{1}{n} \right) \left(2 - \frac{1}{n} \right) \right]$$

$$= 2 \left[1 + \frac{4}{3} \right]$$

$$= \frac{14}{3}$$

Observe that $\int_0^2 (x^2 + 1)\, dx$ is the area of the shaded region in Fig. 7.11.

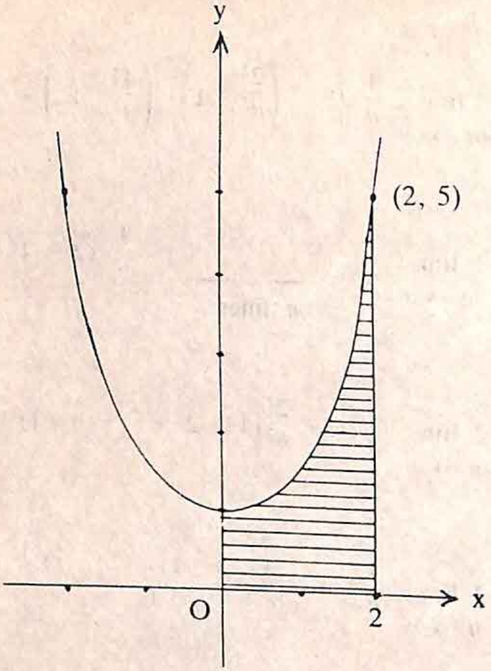

Fig. 7.11

Example 7.3

Evaluate $\int_0^2 e^x\,dx$.

Solution

By definition

$$\int_0^2 e^x\,dx = (2-0)\lim_{n\to\infty}\frac{1}{n}\left[e^0+e^{\frac{2}{n}}+e^{\frac{4}{n}}+\dots+e^{\frac{2n-2}{n}}\right]$$

$$= 2\lim_{n\to\infty}\frac{1}{n}\,\frac{e^{\frac{2n}{n}}-1}{e^{\frac{2}{n}}-1}$$

(using the formula for sum to n terms of a geometric progression)

$$= 2\lim_{n\to\infty}\frac{e^{2}-1}{n\left(e^{\frac{2}{n}}-1\right)} \qquad \dots (7.10)$$

To find this limit, we write down the series for $e^{\frac{2}{n}}$:

$$e^{\frac{2}{n}} = 1+\frac{2}{n}+\frac{1}{2!}\left(\frac{2}{n}\right)^2+\frac{1}{3!}\left(\frac{2}{n}\right)^3+\dots$$

$$\therefore \quad e^{\frac{2}{n}} = \frac{2}{n} + \frac{1}{2!}\left(\frac{2}{n}\right)^2 + \frac{1}{3!}\left(\frac{2}{n}\right)^3 + \dots$$

$$\therefore \quad n\,(e^{\frac{2}{n}} - 1) = 2 + \frac{1}{2!}\left(\frac{4}{n}\right) + \frac{1}{3!}\left(\frac{8}{n^2}\right) + \dots$$

when $n \to \infty$,

$$\left(\frac{1}{2!}\left(\frac{4}{n}\right) + \frac{1}{3!}\left(\frac{8}{n^2}\right) + \dots\right) \to 0$$

[A rigorous proof is not given here.]

$$\therefore \quad \lim_{n \to \infty} n\,(e^{\frac{2}{n}} - 1) = 2.$$

Substituting this in (7.10), we have

$$\lim_{n \to \infty} e^x\,dx = 2 \cdot \frac{e^2 - 1}{2}$$

$$= e^2 - 1.$$

Remark

The above three examples are meant to illustrate the theory. There are, however, simpler methods to calculate the definite integrals. These will be studied in the later sections. This explains why there is less variety of integrands in this section.

Note

(1) In the expression $\int_a^b f(x)\,dx$, a and b are called the *limits of integration*, $f(x)$ is called the *integrand* and x is called the variable of integration.

(2) Our definition makes sense even when $f(x)$ takes negative values. The following points should be noted in that case.

  (i) Some part of the region may come below the x-axis.

  (ii) The area given by $\int_a^b f(x)\,dx$ is now a signed area, in the following sense:

  The area of the portion below the x-axis is given a negative sign. For instance consider $\int_a^b f(x)\,dx$ (See Fig. 7.12).

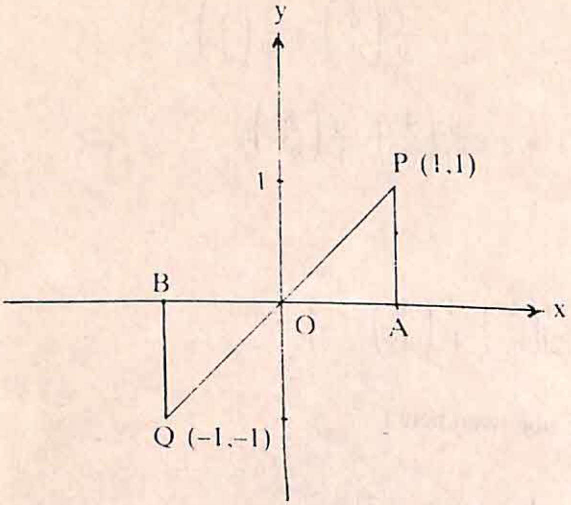

Fig. 7.12

As you can see, the areas of the regions OAP and OBQ are equal. But, since the region OBQ lies below the x-axis, we take its area as $-\frac{1}{2}$, whereas we take the area of the region OAP as $+\frac{1}{2}$. Thus,

$$\int_{1}^{1} f(x)\,dx = \text{algebraic sum of the areas of the regions}$$

$$= \frac{1}{2} + \left(\frac{-1}{2}\right) = 0.$$

EXERCISE 7.1

1. Sketch the regions represented by the following definite integrals:

(a) $\int_0^1 x^2\,dx$ (b) $\int_{-1}^1 (x+3)\,dx$

(c) $\int_1^2 x^3\,dx$ (d) $\int_0^{\frac{\pi}{2}} \sin x\,dx$

(e) $\int_0^{\frac{\pi}{2}} \cos x\,dx$

2. Evaluate the following definite integrals as limit of sums:

(a) $\int_0^5 (x+1)\,dx$ (b) $\int_0^5 (x-1)\,dx$

(c) $\int_{1}^{4} (x^2 - x)\, dx$ (d) $\int_{-1}^{1} e^x\, dx$

3. Compute the following in two methods and verify that you get the same answers. The first method is by using the definition of the direct integral. The second method is by calculating the area enclosed by the trapezium by the known formula.

(a) $\int_{1}^{2} (2x + 3)\, dx$ (b) $\int_{1}^{2} 3x\, dx$

(c) $\int_{3}^{5} (2 - x)\, dx$ (d) $\int_{4}^{\text{[illegible]}} (11x + 1)\, dx$

7.2 Fundamental Theorem of Calculus

Area Function

We have defined $\int_{a}^{b} f(x)\, dx$ as the area of the region bounded by the curve $f(x)$, the x-axis and the ordinates $x = a$, and $x = b$. Let x be a point in $[a, b]$. Then, $\int_{a}^{x} f(x)\, dx$ represents the area of the shaded region in Fig. 7.13.

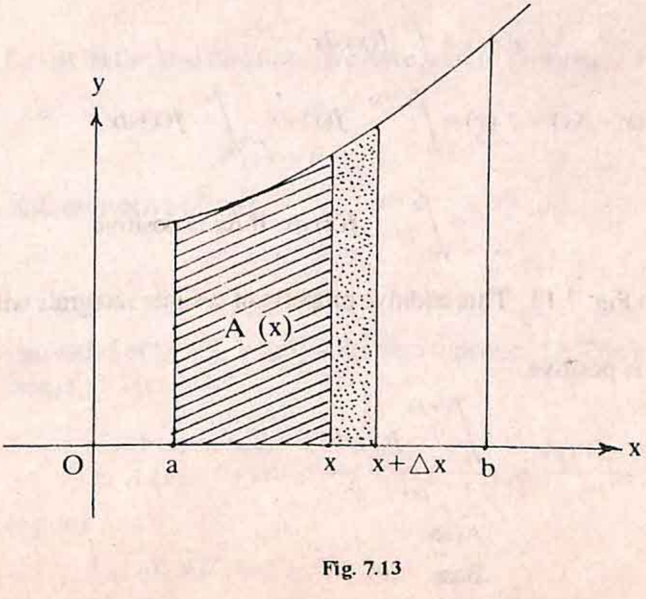

Fig. 7.13

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The area of this shaded region depends on the value of x. In other words the area of this shaded region is a function of x. Let us denote this function of x by $A(x)$. We call the function $A(x)$ as area function.

$$A(x) = \int_a^x f(x)\,dx \qquad \dots (7.11)$$

We will now show that this area function $A(x)$ is an anti-derivative of $f(x)$, i.e., we will show that $A'(x) = f(x)$.

Theorem 7.1 (First Fundamental Theorem of Calculus)

Let the area function be defined by

$$A(x) = \int_a^x f(x)\,dx \text{ for all } x \geq a$$

Then $\qquad A'(x) = f(x)$.

Proof

You will learn the full proof of this theorem in higher classes. Presently we give a proof, assuming a needed result.

Now $\qquad A(x + \Delta x) = \int_a^{x+\Delta x} f(x)\,dx.$

$$A(x) = \int_a^x f(x)\,dx.$$

$$\therefore \quad A(x + \Delta x) - A(x) = \int_a^{x+\Delta x} f(x)\,dx - \int_a^x f(x)\,dx$$

$$= \int_x^{x+\Delta x} f(x)\,dx, \text{ if } \Delta x \text{ is positive}$$

[This is evident from Fig. 7.13. This additive property of definite integrals will be discussed in Section 7.4.]

Therefore, when Δx is positive,

$$\frac{A(x + \Delta x) - A(x)}{\Delta x} = \frac{\int_x^{x+\Delta x} f(x)\,dx}{\Delta x}$$

$$= \frac{\text{Area}}{\text{Base}}$$

= Average value of $f(x)$ in the interval $[x, x + \Delta x]$ ... (7.12)

Similarly, even when Δx is negative, one can prove that (7.11) holds.

Now, we assume without proof, the following result:

If f is a continuous function, then

$$\lim_{\Delta x \to 0} (\text{Average value of } f(x) \text{ between } x \text{ and } x + \Delta x) = f(x) \qquad \ldots (7.13)$$

It follows from (7.12) and (7.13) that

$$\lim_{\Delta x \to 0} \frac{A(x + \Delta x) - A(x)}{\Delta x} = f(x).$$

In other words, $A'(x) = f(x)$.

Theorem 7.2 (Second Fundamental Theorem of Calculus)

Let $F(x)$ be an anti-derivative of $f(x)$. Then

$$\int_a^b f(x)\, dx = F(b) - F(a).$$

Proof

Let $A(x) = \int_a^x f(x)\, dx$ be the area function. We have seen in Theorem 7.1 that $A(x)$ is an anti-derivative of $f(x)$, i.e.

$$A'(x) = f(x) \qquad \ldots (7.14)$$

Given $F(x)$ is an anti-derivative of $f(x)$.

$$\therefore \quad F'(x) = f(x) \qquad \ldots (7.15)$$

$$\therefore \quad A'(x) - F'(x) = 0$$

or in other words derivative of $[A(x) - F(x)]$ is the zero function. By Theorem 6.1, $A(x) - F(x)$ is a constant function, C.

$$\therefore \quad A(x) - F(x) = C$$

$$\text{i.e} \quad A(x) = F(x) + C \qquad \ldots (7.16)$$

When $x = a$, (7.16) gives

$$A(a) = F(a) + C \qquad \ldots (7.17)$$

When $x = b$, (7.16) gives

$$A(b) = F(b) + C \qquad \dots (7.18)$$

From (7.17) and (7.18), we get

$$A(b) - A(a) = F(b) - F(a) \qquad \dots (7.19)$$

Now

$A(a) = \int_a^a f(x)\,dx = 0$ and $A(b) = \int_a^b f(x)\,dx$ (By definition of area function).

Substituting these values in (7.19), we have

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Remark

This theorem is very useful, because it gives us a method of calculating the definite integral (or the area) more easily, without calculating the limit of sums. This is illustrated through the examples below.

Notation

For convenience $F(b) - F(a)$ is denoted by $F(x)\Big]_a^b$, so that

$$\int_a^b f(x)\,dx = F(x)\Big]_a^b,$$

where $F(x)$ is an anti-derivative of $f(x)$.

Steps for calculating $\int_a^b f(x)\,dx$

(i) Find the indefinite integral $\int f(x)\,dx$. Let this be $F(x)$. There is no need to keep the constant C.

(ii) Then take $F(x)\Big]_a^b$. This is $F(b) - F(a)$ and is the value of $\int_a^b f(x)\,dx$.

Example 7.4

Evaluate $\int_1^2 x\,dx$.

Solution

$$\int x\,dx = \frac{x^2}{2}$$

$$\therefore \int_1^2 x\,dx = \frac{x^2}{2}$$

$$= \left(\frac{2^2}{2} - \frac{1^2}{2}\right) = 2 - \frac{1}{2} = \frac{3}{2}.$$

Remark

We solved this very example by taking the limit of sums (See Example 7.1). See how easy the process has become due to the application of the Fundamental Theorem of Calculus.

Example 7.5

Evaluate $\int_0^r \sqrt{r^2 - x^2}\,dx$, where r is a fixed positive number. Hence, prove that the area of a circle of radius r is πr^2.

Solution

We know that

$$\int \sqrt{r^2 - x^2}\,dx = \frac{x}{2}\sqrt{r^2 - x^2} + \frac{r^2}{2}\sin^{-1}\frac{x}{r}$$

(You may work it out.)

$$\therefore \int_0^r \sqrt{r^2 - x^2}\,dx = \left(\frac{x}{2}\sqrt{r^2 - x^2} + \frac{r^2}{2}\sin^{-1}\frac{x}{r}\right]_0^r$$

$$= \frac{r^2}{2}\sin^{-1} 1$$

$$= \frac{r^2}{2}\cdot\frac{\pi}{2}$$

$$= \frac{\pi r^2}{4}$$

Let us sketch the curve $f(x) = y = \sqrt{r^2 - x^2}$

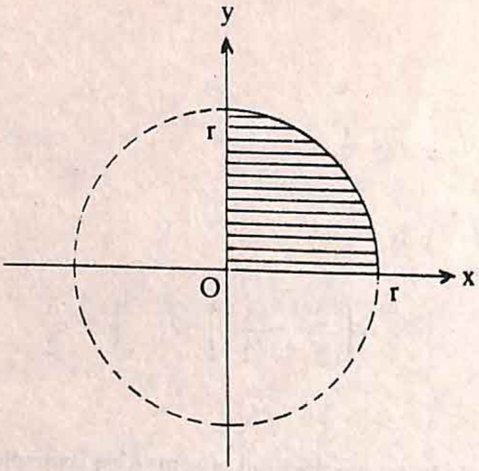


Fig. 7.14

$y=\sqrt{r^2-x^2}$ is equivalent to $y^2=r^2-x^2$, i.e. $x^2+y^2=r^2$, which is a circle with centre at the origin and radius r. Therefore, $\int_0^r \sqrt{r^2-x^2}\,dx$ represents the area bounded by the circle $x^2+y^2=r^2$, the x-axis, the ordinate $x=0$ and the ordinate $x=r$, i.e. it represents the area of the region in the first quadrant as shown in Fig. 7.14.

Hence,

$$\text{the area of the circular region of radius } r = 4\cdot\frac{\pi r^2}{4}$$

$$= \pi r^2.$$

Example 7.6

Evaluate $\int_0^2 \frac{5x+1}{x^2+4}\,dx$.

Solution

First we find the indefinite integral.

$$\int \frac{5x+1}{x^2+4}\,dx = \int \frac{5x}{x^2+4}\,dx + \int \frac{1}{x^2+4}\,dx$$

$$= \int \frac{\frac{5}{2}d\,(x^2+4)}{x^2+4} + \int \frac{1}{x^2+4}\,dx$$

$$= \frac{5}{2}\log(x^2+4) + \frac{1}{2}\tan^{-1}\frac{x}{2}.$$

$$\therefore \int_0^2 \frac{5x+1}{x^2+4}\,dx = \left(\frac{5}{2}\log(x^2+4) + \frac{1}{2}\tan^{-1}\frac{x}{2}\right)\Bigg]_0^2$$

$$= \left(\frac{5}{2}\log 8 + \frac{1}{2}\tan^{-1}1\right) - \left(\frac{5}{2}\log 4 + \frac{1}{2}\tan^{-1}0\right)$$

$$= \left(\frac{5}{2}\log 8 + \frac{1}{2}\cdot\frac{\pi}{4}\right) - \left(\frac{5}{2}\log 4 + \frac{1}{2}\cdot 0\right)$$

$$= \frac{5}{2}\log\frac{8}{4} + \frac{\pi}{8}$$

$$= \frac{5}{2}\log 2 + \frac{\pi}{8}\cdot$$

Example 7.7

Calculate $\int_0^1 (xe^x + \sin\frac{\pi x}{4})\,dx.$

Solution

First we find the indefinite integral of the integrand $\left(xe^x + \sin\frac{\pi x}{4}\right)$ It is $I_1 + I_2$,

where

$$I_1 = \int xe^x\,dx \quad\text{and}\quad I_2 = \int \sin\frac{\pi x}{4}\,dx.$$

Now $\quad I_1 = x\cdot e^x - \int 1\cdot e^x\,dx$ (integrating by parts)

$$= (x-1)\,e^x$$

and $\quad I_2 = \left(-\cos\frac{\pi x}{4}\right)\frac{4}{\pi}$

$$\int\left(xe^x + \sin\frac{\pi x}{4}\right)dx = \left[(x-1)\,e^x - \frac{4}{\pi}\cos\frac{\pi x}{4}\right]_0^1$$

$$= (1-1)\,e^1 - \frac{4}{\pi}\cos\frac{\pi}{4} - \left[(0-1)\,e^0 - \frac{4}{\pi}\cos 0\right]$$

$$= 0 - \frac{4}{\pi}\cos\frac{\pi}{4} - \left(-1 - \frac{4}{\pi}\right)$$

$$= -\frac{4}{\pi}\,\frac{1}{\sqrt{2}} + 1 + \frac{4}{\pi}$$

$$= 1 + \frac{4}{\pi} - \frac{2\sqrt{2}}{\pi}$$

We have now studied both the indefinite integral and the definite integral. In the introductory part of this chapter we said the definite integral is closely related to the indefinite integral. Now you have seen how the two are related. Before we end this section, let us have an overview of these two closely related concepts.

1. The indefinite integral of a function is another function. The definite integral of a function over an interval is a number. The number varies when the interval varies.

2. The indefinite integral $\int f(x)\,dx$ of a function $f(x)$ is not unique. It can vary by a constant.

   The definite integral $\int_a^b f(x)\,dx$ is a unique number.

3. The variable can be denoted by any letter. So it is immaterial whether we write

   $$\int_a^b f(x)\,dx, = \int_a^b f(y)\,dy\,,\ \int_a^b f(t)\,dt, \text{ etc.}$$

   All these symbols denote the same quantity.

4. The definite integral $\int_a^b f(x)\,dx$ is obtainable from indefinite integral $F(x)$ of $f(x)$ by the formula

   $$\int_a^b f(x)\,dx = F(b) - F(a).$$

5. If we know the definite integrals $\int_a^b f(x)\,dx$ for all $b > a$, can we obtain indefinite integral of $f(x)$ from them ? Yes.

   $$A(t) = \int_a^t f(x)\,dx \text{ for all } t \geq a$$

becomes an indefinite integral of $f(x)$.

6. In practical problems, the definite integral occurs more often and the indefinite integral is a tool to find the definite integral.

Relative Merits of the Two Methods

We have thus seen two methods of evaluating definite integrals. The first is the method of limits of sums, studied in Section 7.11. The second is the method using the fundamental theorem, studied in this section. We note the following:

(1) The first method is closer to the geometric meaning. It explains why the definite integral represents the area.

(2) The second method has the following advantages:

(2a) It is easier. For instance in Examples 7.1 and 7.4, we evaluated $\int_0^1 x\,dx$ by these two methods. We noticed that the second method is simpler than the first.

(2b) There are instances where the first method does not work at all, whereas the second method gives the answer. For instance $\int_0^1 xe^x\,dx$ can be found only by the second method.

(2c) There are instances where the first method requires some new formulas whereas the second method gives the answer immediately. For instance, to find $\int_0^{\frac{\pi}{2}} \sin x\; dx$ by the first method, we need a special formula for the sum

$$1 + \sin x + \sin 2x + \ldots + \sin (n-1)\,x.$$

(3) Both the methods give the same answers.

(4) The equivalence of the two methods can be used to evaluate certain limits, using the second method.

Remark

Though a definite integral was evaluated using the knowledge of the corresponding indefinite integral (or anti-derivative) a satisfactory theory requires that the definite integral $\int_a^b f(x)\,dx$ is first defined and the indefinite integral is defined as a function F whereas $F(x) = \int_a^x f(t)\,dt$,

$a \le x \le b$. In fact the formula (Formula 6.1) $\int \frac{dx}{x} = \log|x|$ cannot be used to see whether we can have the definite integral $\int_{-1}^{1} \frac{dx}{x}$ (See p. 269)

EXERCISE 7.2

Evaluate the following definite integrals (1 to 22) in a single step directly:

1. $\int_0^{\frac{\pi}{2}} \cos x \, dx$
2. $\int_0^{\pi} \cos x \, dx$
3. $\int_0^{\frac{\pi}{2}} \cos 2x \, dx$
4. $\int_0^{\frac{\pi}{2}} \sin 2x \, dx$
5. $\int_0^{\frac{\pi}{2}} \sin x \, dx$
6. $\int_0^{\frac{\pi}{2}} (\sin x + \cos x) \, dx$
7. $\int_0^{\frac{\pi}{4}} \sin x \, dx$
8. $\int_4^{5} e^x \, dx$
9. $\int_4^{5} 1 \, dx$
10. $\int_4^{5} 2 \, dx$
11. $\int_{-5}^{5} x \, dx$
12. $\int_{-1}^{2} x \, dx$
13. $\int_{-1}^{1} (x + 1) \, dx$
14. $\int_2^{3} \frac{1}{x} \, dx$
15. $\int_1^{2} (4x^3 - 5x^2 + 6x + 9) \, dx$
16. $\int_0^{\frac{\pi}{4}} \tan x \, dx$

[illegible]. $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot x dx$

18. $\int_0^{\frac{\pi}{4}} \sec x \, dx$

19. $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \operatorname{cosec} x \, dx$

20. $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$

21. $\int_0^1 \frac{dx}{1+x^2}$

22. $\int_2^3 \frac{dx}{x^2-1}$

Evaluate the following definite integrals (23 to 30):

23. $\int_0^{\frac{\pi}{2}} \cos^2 x \, dx$

24. $\int_2^3 \frac{x}{x^2-1} \, dx$

25. $\int_0^1 \frac{2x+3}{5x^2+1} \, dx$

26. $\int_3^4 \frac{dx}{x^2-4} \, dx$

27. $\int_0^1 xe^{x^2} \, dx$

28. $\int_0^{\frac{\pi}{2}} \sqrt{\sin \phi} \, \cos^5 \phi \, d\phi$

29. $\int_1^2 \frac{dx}{(x+1)(x+2)}$

30. $\int_1^2 \frac{5x^2 \, dx}{x^2+4x+3}$

Prove the following (31 to 37):

31. $\int_1^3 \frac{1}{x^2(x+1)} \, dx = \frac{2}{3} + \log \frac{2}{3}$

32. $\int_0^1 xe^x \, dx = 1$

33. $\int_0^{\pi} \frac{d\phi}{5+3\cos\phi} = \frac{\pi}{4}$

34. $\int_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\right) dx = \frac{\pi}{2} - \log 2$

35. $\int_1^{\frac{\pi}{2}} \sin^3 x \, dx = \frac{2}{3}$

36. $\int_0^{\frac{\pi}{4}} 2\tan^3 x \, dx = 1 - \log 2$

37. $\int_0^1 \sin^{-1} x \, dx = \frac{\pi}{2} - 1$

38. If $\int_0^a 3x^2 \, dx = 8$, find the value of a.

39. If $\int_a^b x^3\,dx = 0$ and if $\int_a^b x^2\,dx = \frac{2}{3}$, find both a and b.

40. If $\int_0^a \sqrt{x}\,dx = 2a \int_0^{\frac{\pi}{2}} \sin^3 x\,dx$, find the value of $\int_a^{a+1} x\,dx$.

41. For each of the following functions $f(x)$, verify that

$$\int_0^2 f(x)\,dx = \int_0^1 f(x)\,dx + \int_0^2 f(x)\,dx$$

(a) $f(x) = x + 2$
(b) $f(x) = x^2 + 2$
(c) $f(x) = e^x$
(d) $f(x) = 4x - 5$

42. For each of the following pairs of functions $f(x)$ and $g(x)$, verify that

$$\int_0^1 [f(x) + g(x)]\,dx = \int_0^1 f(x)\,dx + \int_0^1 g(x)\,dx$$

(a) $f(x) = x; g(x) = 1$
(b) $f(x) = 1; g(x) = x^2$
(c) $f(x) = e^x; g(x) = 1$
(d) $f(x) = x^2; g(x) = x^2$

7.3 Evaluation of Definite Integrals by Substitution

The principal step in the evaluation of a definite integral is to find the related indefinite integral. In the previous chapter we have discussed several methods for finding the indefinite integral. of the important methods for finding the indefinite integrals is the method of substitution. One we use substitution method for evaluating the definite integrals, like When

$$\int_0^{\frac{\pi}{4}} \frac{\cos x}{2 + 3\sin x}\,dx, \quad \int_0^1 \frac{2}{(1 + x^2)^2}\,dx$$

the steps could be as follows:

1. Substitute $y = f(x)$ or $x = g(y)$ to reduce the given integral to a known form to integrate. Write the integral in terms of the new variable.

2. Integrate the new integrand with respect to the new variable.
3. Resubstitute for the new variable and write the answer in terms of the original variable x.
4. Find the values of the answers obtained in (3) at the given limits of the integral and find the difference of the values at the upper and lower limits.

In order to quicken this method, we can proceed as follows: After performing steps 1 and 2, there is no need for step 3. Instead, the integral will be kept in the new variable itself, but the limits of the integral will be accordingly changed, so that we can perform the last step. In this section we illustrate this by some examples.

Example 7.8

Evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin 2\phi \, d\phi}{\sin^1\phi + \cos^1\phi}$.

Solution

Let $\quad t = \sin^2 \phi.$

Then $\quad dt = 2 \sin \phi \cos\phi \, d\phi = \sin 2\phi \, d\phi$

and
$$\sin^1 \phi + \cos^1\phi = t^2 + (1-t)^2 = 2t^2 - 2t + 1$$
$$= 2\left(t^2 - t + \frac{1}{2}\right)$$
$$= 2\left[\left(t - \frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2\right]$$

$$\int_0^{\frac{\pi}{2}} \frac{\sin 2\phi \, d\phi}{\sin^1\phi + \cos^1\phi} = \int_a^b \frac{dt}{2\left[\left(t - \frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2\right]}$$

for suitable limits a and b, that are determined as follows:

When ϕ varies from 0 to $\frac{\pi}{2}$, we see that $\sin^2\phi$ varies from 0 to 1. Therefore, 0 and 1 are the limits when t is the variable. The required integral is

$$\int_0^1 \frac{dt}{2\left[\left(t - \frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2\right]} = \frac{1}{2} \, \frac{1}{\frac{1}{2}} \tan^{-1} \left.\frac{t - \frac{1}{2}}{\frac{1}{2}}\right]_0^1$$

$$= \tan^{-1}(2t-1)\Big]_0^1$$

$$= \tan^{-1} 1 - \tan^{-1}(-1)$$

$$= \frac{\pi}{4} - \left(-\frac{\pi}{4}\right)$$

$$= \frac{\pi}{2}$$

Example 7.9

Evaluate $\displaystyle\int_0^{\pi} \frac{dx}{5 + 4\cos x}$.

Solution

Let $\qquad t = \tan \dfrac{x}{2}$.

Then $\qquad dt = \dfrac{1}{2}(1 + t^2)\, dx$

and $\qquad \cos x = \dfrac{1 - t^2}{1 + t^2}$.

When $x = 0$, $t = 0$.
When $x = \pi$, t tends to ∞.
As x varies from 0 to π, t varies from 0 to ∞.

$$\therefore \quad \int_0^{\pi} \frac{dx}{5 + 4\cos x} = \int_0^{\infty} \frac{2\,dt}{(1 + t^2)\left[5 + 4\left(\frac{1 - t^2}{1 + t^2}\right)\right]}$$

$$= \int_0^{\infty} \frac{2\,dt}{5(1 + t^2) + 4(1 - t^2)}$$

$$= 2\int_0^{\infty} \frac{dt}{t^2 + 9}$$

$$= \frac{2}{3}\tan^{-1}\frac{t}{3}\Big]_0^{\infty}$$

$$= \frac{2}{3}\left(\frac{\pi}{2} - 0\right) = \frac{\pi}{3}$$

EXERCISE 7.3

Evaluate the following integrals by changing the limits of integration after suitable substitutions:

1. $\int_2^3 \frac{x}{x^2+1}\,dx$
2. $\int_0^1 \frac{2x+3}{5x^2+1}\,dx$
3. $\int_0^1 xe^{x^2}\,dx$
4. $\int_0^{\frac{\pi}{2}} \sqrt{\sin\phi}\,\cos^5\phi\,d\phi$
5. $\int_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx$
6. $\int_0^2 x\sqrt{x+2}\,dx$
7. $\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x}\,dx$
8. $\int_0^{\frac{\pi}{2}} \frac{dx}{2\cos x + 4\sin x}$
9. $\int_0^{\frac{\pi}{2}} \frac{dx}{a\cos x + b\sin x}$, $a, b > 0$
10. $\int_0^{\frac{\pi}{2}} \frac{dx}{5+4\sin x}$
11. $\int_0^2 \frac{dx}{x+4-x^2}$
12. $\int_0^a \frac{dx}{ax+a^2-x^2}$
13. Prove that $\int_0^{\pi} \frac{dx}{1+\sin x} = 2$

7.4 Some Properties of Definite Integrals

We list below some important properties of definite integrals. These will be useful in evaluating the definite integrals more easily.

1. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

2. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

3. $\int_a^a f(x)\,dx = 0$

4. $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

5. $\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx$ if $f(x)$ is an even function.

6. $\int_{-a}^a f(x)\,dx = 0$ if $f(x)$ is an odd function.

We explain these one by one now.

Formula 7.1

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

Explanation

We have defined $\int_a^b f(x)\,dx$, only when $a < b$. But in practice there are occasions to consider $\int_b^a f(x)\,dx$ even when a is not less than b. In such occasions, we take $\int_a^b f(x)\,dx$ to mean $\int_b^a f(x)\,dx$, noting that b is less than a.

Remark

In order to compute $\int_a^b f(x)\,dx$ where $a < b$, sometimes we make a substitution $x = g(y)$. Then the integral becomes $\int_{g(a)}^{g(b)} f(g(y))\,g(y)\,dy$. Now there is no guarantee that $g(a) < g(b)$. Thus, there are occasions for considering $\int_a^b f(x)\,dx$ where $b < a$.

Remark

Let $F(x)$ be an anti-derivative of $f(x)$. Then we have seen in Section 7.2 that

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Note that this is consistent with the formula

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

because

$$F(b) - F(a) = -[F(a) - F(b)].$$

In other words, because of the formula 7.1, the result

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

holds even when $b < a$.

Formula 7.2

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx,$$

where a, b, c are any three real numbers.

Explanation

(i) When $a < c < b$, this is explained by Fig. 7.15.

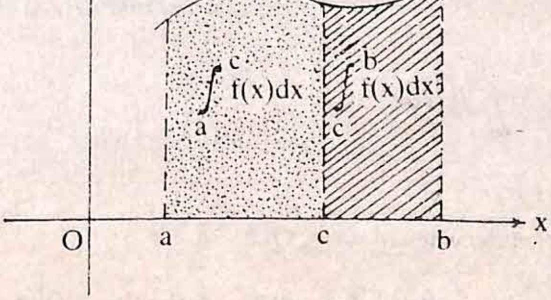


Fig. 7.15

The area enclosed between

$$\begin{aligned} x &= a \\ x &= b \\ y &= 0 \\ \text{and} \quad y &= f(x) \end{aligned}$$

is the sum of the two areas differently shaded in the figure.

(ii) When c is not between a and b, then also the same formula is true. Suppose $a < b < c$. Then by the first case,

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$

$$\therefore \quad \int_a^b f(x)\,dx = \int_a^c f(x)\,dx - \int_b^c f(x)\,dx$$

$$= \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \text{ (using Formula 7.1).}$$

Similarly, the same formula can be proved when $c < a < b$.

Remark

Formula 7.2 is consistent with the result

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is an anti-derivative of f and a, b are any two real numbers. This is because

$$F(b) - F(a) = [F(b) - F(c)] + [F(c) - F(a)]$$

holds for all real numbers a, b, c.

Formula 7.3

$$\int_a^a f(x)\,dx = 0$$

Explanation

(i) We have defined $\int_a^b f(x)\,dx$, when $a < b$ as

$$(b - a) \cdot \lim_{n \to \infty} \frac{f(a) + f(a + h) + \dots + f[a + (n - 1)h]}{n}$$

where $h = \dfrac{b-a}{n}$

If we want the same formula to hold even when $a = b$, we must have $\int_a^a f(x)\,dx = 0$ because $a - a$ occurs as a factor.

(ii) This is also in conformity with the geometric intution that the area of a line segment is zero.

(iii) This is also consistent with the formula

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is an anti-derivative of f. This is because $F(b) - F(a) = 0$ when $b = a$.

Formula 7.4

$$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$$

Explanation

When the lower limit of the integral is zero, the integrand $f(x)$ can be changed as $f(a-x)$, where a is the upper limit of the integral, without affecting the value of the integral.

Proof

Let $y = a - x$.

Then $dy = -\,dx$

When $x = 0$, $y = a$.

When $x = a$, $y = 0$.

As x varies from 0 to a, y varies from a to 0.

Also $x = a - y$.

$$\therefore \quad \int_0^a f(x)\,dx = \int_a^0 f(a-y)\,(-dy)$$

$$= -\int_a^0 f(a-y)\,dy$$

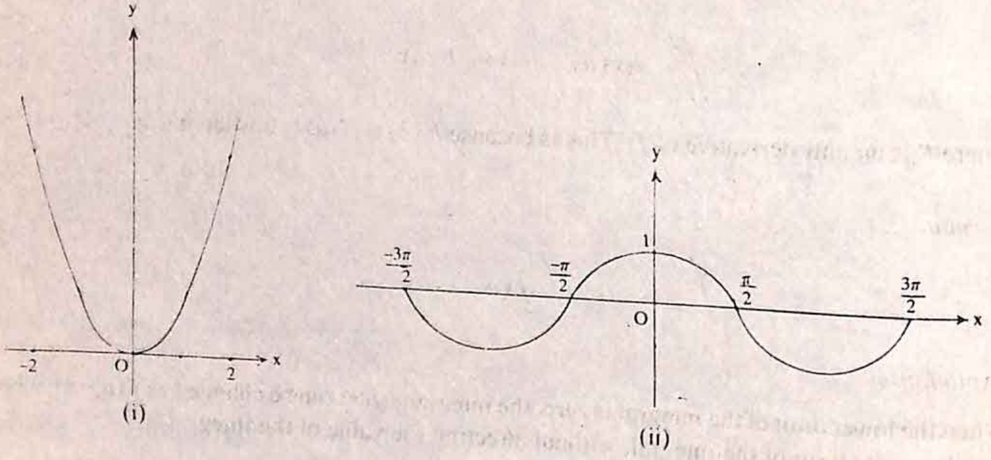
$$= \int_0^a f(a-y)\,dy \qquad \text{(by Formula 7.1)}$$

$$= \int_0^a f(a-x)\,dx$$

(by changing the variable y to x)

Remark

This is a very important property, because it is useful in evaluating many definite integrals as seen in the examples that follow:



Graph of x^2 on $[-2, 2]$

Graph of $\cos x$ on $\left[-\frac{3\pi}{2}, \frac{3\pi}{2}\right]$

Graphs of even functions
Fig. 7.16

To explain the last two properties in the above list we recall two definitions.

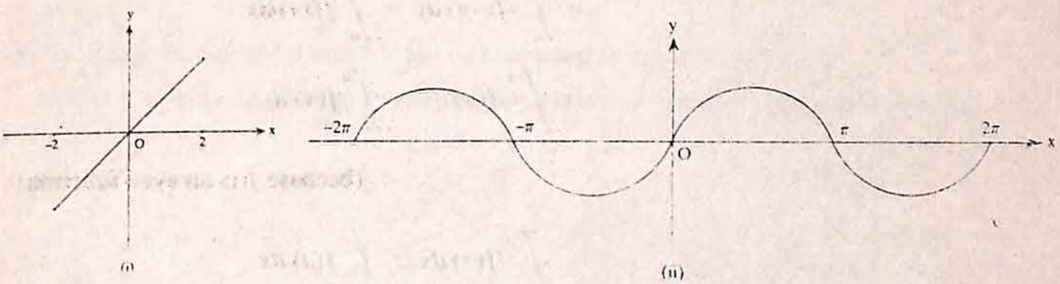
Definition

A real function on an interval of the form $[-a, a]$ is said to be an even function if $f(-x) = f(x)$ holds for all x in the interval.

For example, x^2, $\cos x$, $\sin^2 x$ are all even functions on any interval $[-a, a]$ of R. Their graphs are symmetric about the y-axis. See the graphs of x^2 and $\cos x$ in Fig. 7.16.

Definition

A real function on an interval of the form $[-a, a]$ is said to be an odd function if $f(-x) = -f(x)$ holds for all x in the interval. For example, x, $2x^3$, [illegible] $\sin x$ are all odd functions. Their graphs are symmetric with respect to the origin (See Fig. 7.17).



Graph of x on $[-2, 2]$

Graph of $\sin x$ on $[-2\pi, 2\pi]$

Graphs of odd functions
Fig. 7.17

Formula 7.5

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

if $f(x)$ is an even function.

Proof

Let $f(x)$ be an even function. Then

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx$$

$$= \int_{a}^{0} f(-y)\,(-dy) + \int_{0}^{a} f(x)\,dx$$

(by putting $y = -x$ in the first of the two integrals)

$$= -\int_{a}^{0} f(-y)\,dy + \int_{0}^{a} f(x)\,dx$$

$$= \int_0^a f(-y)\,dy + \int_0^a f(x)\,dx$$

$$= \int_0^a f(y)\,dy + \int_0^a f(x)\,dx$$

(because f is an even function)

$$= \int_0^a f(x)\,dx + \int_0^a f(x)\,dx$$

(because y may be replaced by x throughout)

$$= 2\int_0^a f(x)\,dx\,.$$

Formula 7.6

$$\int_{-a}^a f(x)\,dx = 0\,,$$

if $f(x)$ is an odd function.

Proof

Let $f(x)$ be an odd function. Then

$$\int_{-a}^a f(x)\,dx = \int_{-a}^0 f(x)\,dx + \int_0^a f(x)\,dx$$

$$= \int_0^a f(-y)\,dy + \int_0^a f(x)\,dx$$

(by putting $y = -x$ in the first of the two integrals)

$$= \int_0^a -f(y)\,dy + \int_0^a f(x)\,dx$$

(because f is an odd function)

$$= -\int_0^a f(y)\,dy + \int_0^a f(x)\,dx$$

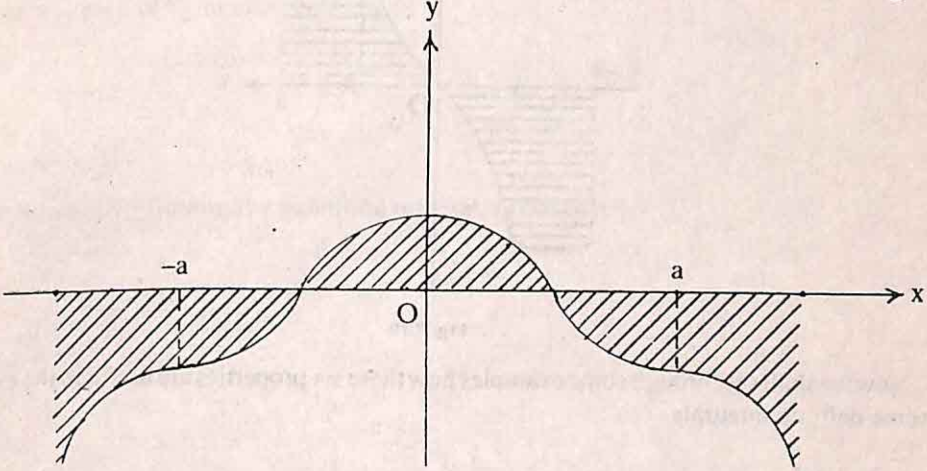
$$= -\int_0^a f(x)\,dx + \int_0^a f(x)\,dx$$

$$= 0.$$

Remark

Geometrically, formulas 7.5 and 7.6 can be explained as under:

If $f(x)$ is an even function on $[-a, a]$, its graph is symmetric about the y-axis (See Fig. 7.18)



Graph of an even function
Fig. 7.18

Therefore, the area represented by $\int_{-a}^{a} f(x)\,dx$ is equally divided on the left and right sides of the y-axis.

$\therefore$ The total area = twice the area to the right of the y-axis

This is what the formula

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

means.

On the other hand, if $f(x)$ is an odd function (Fig. 7.19), the area represented by $\int_{-a}^{a} f(x)\,dx$, is divided by the y-axis into two parts equal in magnitude but opposite in sign. (Recall that the area is considered negative if the region is below the x-axis). Therefore, the total signed area is zero. This is what the Formula 7.6 says.

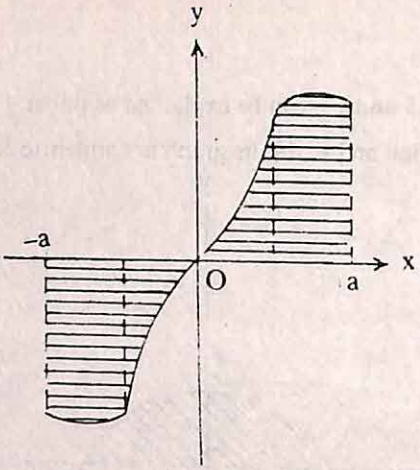


Fig. 7.19

Now we shall see through some examples how these six properties are useful in the evaluation of some definite integrals.

Example 7.10

Evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}\, dx}{\sqrt{\sin x}+\sqrt{\cos x}}$

Solution

Let $$f(x) = \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}$$

Then $$f\left(\frac{\pi}{2}-x\right) = \frac{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}}{\sqrt{\sin\left(\frac{\pi}{2}-x\right)} + \sqrt{\cos\left(\frac{\pi}{2}-x\right)}}$$

$$= \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}}$$

We note that $$f(x) + f\left(\frac{\pi}{2}-x\right) = \frac{\sqrt{\sin x} + \sqrt{\cos x}}{\sqrt{\sin x}\quad \sqrt{\cos x}} = 1.$$

$$\therefore \int_0^{\frac{\pi}{2}} f(x)\, dx + \int_0^{\frac{\pi}{2}} f\left(\frac{\pi}{2}-x\right) dx = \int_0^{\frac{\pi}{2}} 1\, dx = \Big[\, x \,\Big]_0^{\frac{\pi}{2}} = \frac{\pi}{2}$$

But we also know that

$$\int_0^{\frac{\pi}{2}} f(x)\,dx = \int_0^{\frac{\pi}{2}} f\left(\frac{\pi}{2} - x\right) dx \quad \text{(by formula 7.4)}$$

Therefore, each of them must be equal to $\frac{\pi}{4}$.

Thus

$$\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}\,dx}{\sqrt{\sin x} + \sqrt{\cos x}} = \frac{\pi}{4}.$$

Example 7.11

Without actually finding the indefinite integral, prove that

$$\int_0^{\frac{\pi}{2}} \sin^2 x\,dx = \frac{\pi}{4}.$$

Solution

Let

$$I = \int_0^{\frac{\pi}{2}} \sin^2 x\,dx.$$

Then

$$I = \int_0^{\frac{\pi}{2}} \sin^2\left(\frac{\pi}{2} - x\right) dx \text{ (by formula 7.4)}$$

$$= \int_0^{\frac{\pi}{2}} \cos^2 x\,dx.$$

$$\therefore \quad I + I = \int_0^{\frac{\pi}{2}} (\sin^2 x + \cos^2 x)\,dx = \int_0^{\frac{\pi}{2}} 1\,dx = \frac{\pi}{2}.$$

It follows that

$$I = \frac{\pi}{4}.$$

Remark

Sometimes these formulas yield quicker results for some problems that can be solved otherwise also.

Example 7.12

Calculate $\int_0^2 x\sqrt{2-x}\,dx$.

Solution

First Method

Let $y = \sqrt{2-x}$.

Then $y^2 = 2 - x$.

$\therefore \quad 2y\,dy = -\,dx$

Also $x = 2 - y^2$.

When $x = 0$, $y = \sqrt{2}$.

When $x = 2$, $y = 0$.

$$\therefore \int_0^2 x\sqrt{2-x}\;dx = \int_{\sqrt{2}}^{0} (2-y^2)\,y\,(-2y\,dy)$$

$$= \int_0^{\sqrt{2}} 2y^2(2-y^2)\,dy \quad \text{(by Formula 7.1)}$$

$$= \left[\frac{4y^3}{3} - \frac{2y^5}{5}\right]_0^{\sqrt{2}}$$

$$= \frac{8\sqrt{2}}{3} - \frac{8\sqrt{2}}{5}$$

$$= \frac{16\sqrt{2}}{15}.$$

Second Method

$$\int_0^2 x\;\sqrt{2-x}\;dx = \int_0^2 (2-x)\,\sqrt{2-(2-x)}\;dx \qquad \text{(by formula 7.4)}$$

$$= \int_0^2 (2-x)\,\sqrt{x}\;dx$$

$$= \int_0^2 (2\sqrt{x} - x\,\sqrt{x}\,)\,dx$$

$$= \left[\frac{2x^{\frac{3}{2}}}{\frac{3}{2}} \quad \frac{x^{\frac{5}{2}}}{\frac{5}{2}}\right]_0^2$$

$$= \frac{8\sqrt{2}}{3} - \frac{8\sqrt{2}}{5}$$

$$= \frac{16\sqrt{2}}{15}$$

Example 7.13

Evaluate $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin^2 x \, dx$.

Solution

We observe that $\sin^2 x$ is an even function.

Therefore
$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin^2 x \, dx = 2\int_0^{\frac{\pi}{4}} \sin^2 x \, dx$$

$$= 2\int_0^{\frac{\pi}{4}} \frac{1 - \cos 2x}{2} \, dx$$

$$= \int_0^{\frac{\pi}{4}} (1 - \cos 2x) \, dx$$

$$= \left[x - \frac{1}{2} \sin 2x \right]_0^{\frac{\pi}{4}}$$

$$= \left(\frac{\pi}{4} - \frac{1}{2} \sin \frac{\pi}{2} \right) - 0$$

$$= \frac{\pi}{4} - \frac{1}{2}.$$

Example 7.14

Evaluate $\int_{-5}^{5} |x + 2| \, dx$.

Solution

$$|x+2| = \begin{cases} x+2 & \text{if } x \geq -2 \\ -x-2 & \text{if } x < -2 \end{cases}$$

$$\therefore \int_{-5}^{5} |x+2| = dx \int_{-5}^{-2} |x+2|\, dx + \int_{-2}^{5} |x+2|\, dx \qquad \text{(by formula 7.2)}$$

$$= \int_{-5}^{-2} (-x-2)\, dx + \int_{-2}^{5} (x+2)\, dx$$

$$= \left[-\frac{x^2}{2} - 2x\right]_{-2}^{-5} + \left[\frac{x^2}{2} + 2x\right]_{-2}^{5}$$

$$= -\frac{4}{2} + 4 - \left(-\frac{25}{2} + 10\right) + \frac{25}{2} + 10 - \left(\frac{4}{2} - 4\right)$$

$$= 4 + \frac{25}{2} + \frac{25}{2} \qquad \text{(since all other terms cancel mutually)}$$

$$= 29.$$

Example 7.15

For every function $f(x)$, prove that

$$\int_{0}^{2a} f(x)\, dx = \int_{0}^{a} f(x)\, dx + \int_{0}^{a} f(2a-x)\, dx.$$

Solution

We know

$$\int_{0}^{2a} f(x)\, dx = \int_{0}^{a} f(x)\, dx + \int_{0}^{2a} f(x)\, dx \qquad \text{(by Formula 7.2)}$$

Therefore, it is enough to prove that

$$\int_{0}^{2a} f(x)\, dx = \int_{0}^{a} f(2a-x)\, dx.$$

For this let $y = 2a - x.$

Then $dy = -\, dx.$

When $x = a, y = a$.

When $x = 2a, y = 0$.

Also $x = 2a - y$.

$$\therefore \int_0^{2a} f(x)\,dx = \int_0^{a} -f(2a-y)\,dy$$

$$= \int_0^{a} f(2a-y)\,dy \quad \text{(by Formula 7.1)}$$

$$= \int_0^{a} f(2a-x)\,dx.$$

Example 7.16

If $f(2a - x) = f(x)$ for all x, prove that

$$\int_0^{2a} f(x)\,dx = 2\int_0^{a} f(x)\,dx.$$

Solution.

$$\int_0^{2a} f(x)\,dx = \int_0^{a} f(x)\,dx + \int_0^{a} f(2a-x)\,dx \qquad \text{(by Example 7.15)}$$

$$= \int_0^{a} f(x)\,dx + \int_0^{a} f(x)\,dx \quad [\text{because } f(2a-x) = f(x)]$$

$$= 2\int_0^{a} f(x)\,dx.$$

EXERCISE 7.4

By using the properties of definite integrals, evaluate the following:

1. $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin^{\frac{3}{2}} x\,dx}{(\sin^{\frac{3}{2}} x + \cos^{\frac{3}{2}} x)}$

2. $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\cos^5 x\,dx}{(\sin^5 x + \cos^5 x)}$

3. $\int_0^{\frac{\pi}{4}} \log(1+\tan x)\, dx$

4. $\int_0^1 x(1-x)^n\, dx$

5. $\int_0^{\frac{\pi}{2}} (2\log\sin x - \log\sin 2x)dx$

6. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x\, dx$

7. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^7 x\, dx$

8. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 x\, dx$

9. $\int_0^{\pi} \frac{x\, dx}{1+\sin x}$

10. $\int_0^{\frac{\pi}{2}} \frac{x\, dx}{\sin x+\cos x}$

11. $\int_0^{\frac{\pi}{2}} \log\tan x\, dx$

12. $\int_2^8 |x-5|\, dx$

13. $\int_1^4 f(x)\, dx$ where

$$f(x) = \begin{cases} 7x+3 & \text{if } 1 \le x \le 3 \\ 8x & \text{if } 3 \le x \le 4 \end{cases}$$

14. $\int_0^9 f(x)\, dx$ where

$$f(x) = \begin{cases} \sin x \text{ if } 0 \le x \le \frac{\pi}{2} \\ 1 \text{ if } \frac{\pi}{2} \le x \le 3 \\ e^{x-3} \text{ if } 3 \le x \le 9 \end{cases}$$

15. $\int_0^{2\pi} \cos^5 x\, dx$

Prove the following (16 to 21):

16. $\int_0^{\pi} \frac{x\tan x}{\sec x \operatorname{cosec} x}\, dx = \frac{\pi^2}{4}$

17. $\int_0^{\pi} \frac{x\tan x}{\sec x+\tan x}\, dx = \pi\left(\frac{\pi}{2}-1\right)$

18. $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin x - \cos x}{1 + \sin x \cos x}\, dx = 0$

19. $\displaystyle\int_0^1 \log\left(\frac{1}{x} - 1\right) dx = 0$

20. $\displaystyle\int_0^{\frac{\pi}{2}} \sin 2x \log \tan x \, dx = 0$

21. $\displaystyle\int_0^{\pi} \frac{dx}{1 + \sin x} = 2$

22. Evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin^{-1} x \, dx}{\sin^{-1} x + \cos^{-1} x}$

23. Evaluate $\displaystyle\int_0^{a} \frac{x\, dx}{(1+x)(1+x^2)}$

24. Evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \sin 2x \tan^{-1}(\sin x)\, dx$

25. Evaluate $\displaystyle\int_1^{a} \frac{dx}{x^2(1+x)}$

26. Evaluate $\displaystyle\int_0^{\frac{\pi}{4}} \frac{(\sin x + \cos x)}{9 + 16 \sin 2x}\, dx$

27. Evaluate $\displaystyle\int_{-1}^{\frac{3}{2}} |x \sin \pi x|\, dx.$

7.5 Areas of Bounded Regions

In this section, we use integration to compute areas of certain regions. Some problems of this nature have been already met in earlier sections. For instance, we have calculated the areas of circular regions in Section 7.2.

Example 7.17

Find the area of the region bounded by the ellipse

$$\frac{x^2}{4} + \frac{y^2}{9} = 1.$$

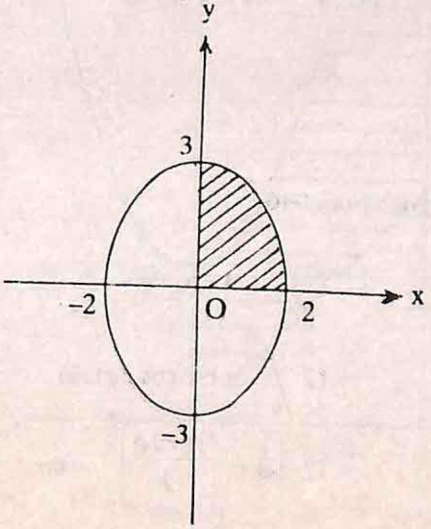


Fig. 7.20

Solution

We rewrite the equation in the form

$$y^2 = 9\left(1 - \frac{x^2}{4}\right)$$

We observe the symmetry of the ellipse about the x-axis and the y-axis (Fig. 7.20). Therefore, the required area is 4 times the area in the first quadrant. In the first quadrant

$$y = 3\sqrt{1 - \frac{x^2}{4}} = \frac{3}{2}\sqrt{4 - x^2}\,.$$

The limits of the integral are $x = 0$ and $x = 2$. (Note that every point inside the ellipse has x-coordinate not exceeding 2.) Therefore, to find the above area we have to compute

$$\frac{3}{2}\int_0^2 \sqrt{4 - x^2}\,dx$$

and then multiply it by 4, i.e. we have to compute

$$A = 6\int_0^2 \sqrt{4 - x^2}\,dx$$

Put $x = 2\sin\phi.$

Then $dx = 2\cos\phi\,d\phi.$

$$\therefore\ \sqrt{4 - x^2} = \sqrt{4 - 4\sin^2\phi} = 2\cos\phi.$$

When $x = 0,\ \phi = 0.$

When $x = 2,\ \phi = \frac{\pi}{2}.$

As x varies from 0 to 2, ϕ varies from 0 to $\frac{\pi}{2}$.

$$\therefore\quad A = 6\int_0^2 \sqrt{4 - x^2}\,dx$$

$$= 12\int_0^{\frac{\pi}{2}} (1 + \cos 2\phi)\,d\phi$$

$$= 12\left[\phi + \frac{\sin 2\phi}{2}\right]_0^{\frac{\pi}{2}} = 6\pi.$$

Therefore, area of the region is 6π.

Example 7.18

Find the area of the region bounded by the curves

$$y = x^2 + 2,$$
$$y = x,$$
$$x = 0$$
$$\text{and} \quad x = 3.$$

Also sketch the region bounded by these curves.

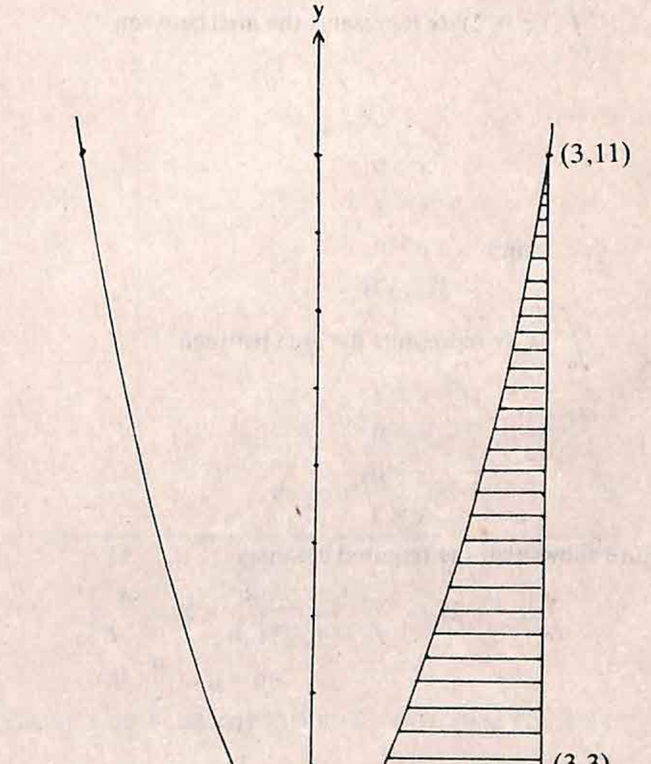


Fig. 7.21

Solution

$y = x^2 + 2$ is the equation of a parabola.

$y = x$ is the equation of a straight line lying below the parabola.

(You can prove that $x^2 + 2 > x$ for all numbers x.)

The region specified in the problem is as in Fig. 7.21.

$\therefore \int_0^3 (x^2 + 2)\, dx$ represents the area between

$$y = x^2 + 2,$$
$$x = 0$$
$$x = 3$$
$$\text{and} \quad y = 0.$$

$\therefore \int_0^3 x\, dx$ represents the area between

$$y = x,$$
$$y = 0,$$
$$x = 0$$
$$\text{and} \quad x = 3.$$

A look at the figure shows that the required area is

$$\int_0^3 (x^2 + 2)\, dx - \int_0^3 x\, dx = \left[\frac{x^3}{3} + 2x - \frac{x^2}{2}\right]_0^3$$
$$= 9 + 6 - \frac{9}{2} - 0$$
$$= \frac{21}{2}$$

Example 7.19

Using integration find the area of the region bounded by the triangle whose vertices are (1, 0), (2, 2) and (3, 1).

Solution

Let the vertices be A, B, C as shown in Fig. 7.22

The equation of AB is

$$y = 2x - 2.$$

The equation of BC is

$$y = -x + 4.$$

This leads us to consider

$$f(x) = \begin{cases} 2x - 2 & \text{if } 1 \leq x \leq 2 \\ 4 - x & \text{if } 2 < x \leq 3 \end{cases}$$

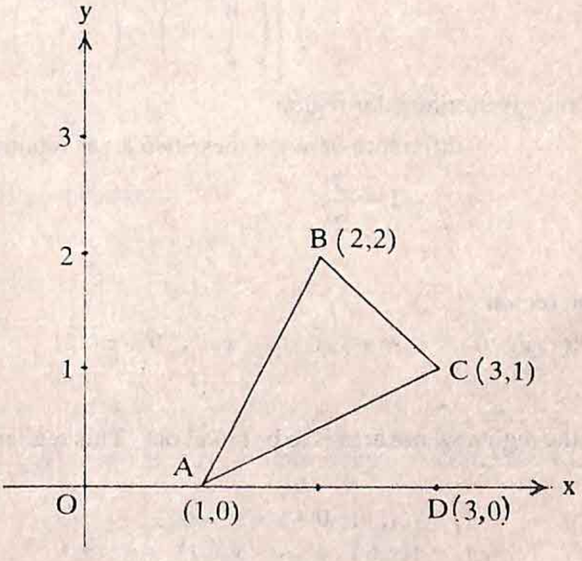


Fig. 7.22

Now, $\int_1^3 f(x)\,dx$ gives the area of the quadrilaterial $ABCD$ (See Fig. 7.22). This area is

$$\begin{aligned} \int_1^3 f(x)\,dx &= \int_1^2 f(x)\,dx + \int_1^3 f(x)\,dx \\ &= \int_1^2 (2x-2)\,dx + \int_2^3 (4-x)\,dx \\ &= \Big[x^2 - 2x\Big]_1^2 + \left[4x - \frac{x^2}{2}\right]_2^3 \\ &= [(4-4)-(1-2)] + \left[\left(12 - \frac{9}{2}\right) - (8-2)\right] \end{aligned}$$

$$= \frac{5}{2}.$$

Next, the equation of AC is

$$y = \frac{1}{2}(x-1).$$

$\therefore$ the area of the triangle ACD is given by

$$\int_1^3 \frac{1}{2}\,(x-1)\,dx = \frac{1}{2}\left(\frac{x^2}{2} - x\right]_1^3$$

$$= \frac{1}{2}\left[\left(\frac{9}{2} - 3\right) - \left(\frac{1}{2} - 1\right)\right] = 1$$

Lastly, the area of the given triangular region

= difference between these two areas found above

$$= \frac{5}{2} - 1 = \frac{3}{2}$$

Example 7.20

Find the area of the region

$$\{(x, y) : 0 \le y \le x^2 + 1,\ 0 \le y \le x + 1,\ 0 \le x \le 2\}.$$

Solution

Let us first sketch the region whose area is to be found out. This region is the intersection of the following regions:

$$A_1 = \{(x, y) : 0 \le y \le x^2 + 1\},$$
$$A_2 = \{(x, y) : 0 \le y \le x + 1\}$$
and $$A_3 = \{(x, y) : 0 \le x \le 2\}.$$

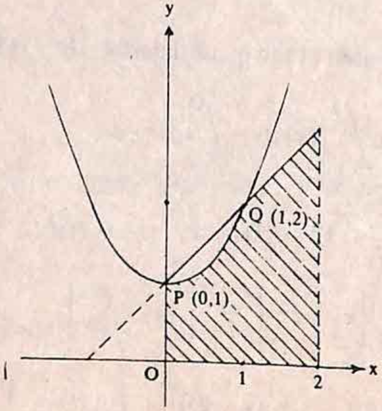


Fig. 7.23

The function for which the curve in Fig. 7.23 is the graph is given by

$$f(x) = \begin{cases} x^2 + 1 & \text{if } 0 \le x \le 1 \\ x + 1 & \text{if } 1 < x \le 2 \end{cases}$$

The points of intersection of

$$y = x^2 + 1$$

$$\text{and} \qquad y = x + 1$$

are $P(0, 1)$ and $Q(1, 2)$. Thus,

$$f(x) = \begin{cases} x^2 + 1 & \text{for } 0 \le x \le 1 \\ x + 1 & \text{for } 1 < x \le 2 \end{cases}$$

The region considered is bounded by

$$y = f(x),$$

$$y = 0,$$

$$x = 0$$

$$\text{and} \qquad x = 2.$$

The required area is $\int_0^2 f(x)\,dx$ for this $f(x)$. This integral is computed as follows:

$$\int_0^2 f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dy$$

$$= \int_0^1 (x^2 + 1)\,dx + \int_1^2 (x + 1)\,dx$$

$$= \left(\frac{x^3}{3} + x\right)\Bigg]_0^1 + \left(\frac{x^2}{2} + x\right)\Bigg]_1^2$$

$$= \left[\left(\frac{1}{3} + 1\right) - (0)\right] + \left[(2 + 2) - \left(\frac{1}{2} + 1\right)\right]$$

$$= \frac{23}{6}.$$

Example 7.21

Find the area of the region bounded by

$$y = -1$$
$$y = 2,$$
$$x = y^3$$
and $$x = 0.$$

Solution

Here $x = y^3$ is not in the form $y = f(x)$. Therefore, we interchange x and y throughout and consider the region bounded by

$$x = -1,$$
$$x = 2,$$
$$y = x^3$$
and $$y = 0.$$

You may verify that these two regions have the same area. Its area is given by

$$\int_{-1}^{2} x^3\,dx = \left[\frac{x^1}{4}\right]_1^2$$
$$= \frac{16}{4} - \frac{1}{4} = \frac{15}{4}.$$

EXERCISE 7.5

1. Find the area of the region
$$\{(x, y) : x^2 \le y \le x\}.$$
2. Find the area of the region
$$\{(x, y) : x^2 \le y \le |x|\}.$$
3. Find the area of the region
$$\{(x, y) : x^2 + y^2 \le 1 \le x + y\}.$$
4. Using integration find the area of the triangular region whose sides have the equation
$$y = 2x + 1,$$
$$y = 3x + 1,$$
and $$x = 4.$$

5. Find the area of the region bounded by the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

6. Find the area of the region between the curves

$$y^2=4x$$

and $$x=3.$$

7. Find the area of the region

$$\{(x, y) : y^2 \le 4x,\ 4x^2+4y^2 \le 9\}$$

8. Using integration, find the area of the region bounded by the triangle whose vertices are $(-1, 1)$, $(0, 5)$ and $(3, 2)$

9. Find the area of the region enclosed by the parabola $y^2 = 4ax$ and the chord $y = mx$.

10. Calculate the area of the region bounded by the two parabolas $y = x^2$ and $x = y^2$.

11. Make a rough sketch of the curve $y = 2\sin x$, where $0 \le x \le \pi$ and determine the area of the region enclosed between the curve and the x-axis.

12. Find the area of the region enclosed between the two circles

$$x^2+y^2=1 \text{ and } (x-1)^2+y^2=1.$$

13. Find the area of the region bounded by the curve $y^2 = 2y - x$ and the y-axis.
(Hint: This is same as the area bounded by the curve $y = 2x - x^2$ and the x-axis.)

14. Find the area of the region bounded by the parabola $y^2 = 4ax$ and its latus rectum.

15. Find the area bounded by the curve $x^2 = 4y$ and the straight line $x = 4y - 2$.

16. Find the area of the smaller region bounded by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the straight line $\frac{x}{a}+\frac{y}{b}=1$.

CHAPTER 8

Differential Equations

8.1 Introduction

In many physical situations the relation between the rates of change of observable quantities is simpler than the relation between the quantities themselves. For this reason differential equations arise naturally as models for several problems in engineering, physical science, biological sciences and social sciences. The subject of differential equations is vast and is also of great current research interest. In this chapter we shall be learning some basic concepts. We shall also consider some physical and geometrical problems which give rise to differential equations. Finally, we shall discuss some methods of solving differential equations.

8.2 Differential Equations: Order and Degree

As stated in the introduction many important problems in engineering, physical sciences and social sciences, when formulated in mathematical terms, lead to equations that involve diffferentials or differential coefficients (derivatives). Such equations are called differential equations. The following are a few examples of differential equations:

$$\frac{dy}{dx} = \cos x \qquad \dots (8.1)$$

$$\frac{d^2y}{dx^2} + y = 0 \qquad \dots (8.2)$$

$$dx + dy = 0 \qquad \dots (8.3)$$

$$x\,dx + y\,dy = 0 \qquad \dots (8.4)$$

$$\left(\frac{d^2y}{dx^2}\right)^2 + x^2\left(\frac{dy}{dx}\right)^3 = 0 \qquad \dots (8.5)$$

$$y = \frac{dy}{dx} + \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \qquad \dots (8.6)$$

$$\frac{d^2y}{dx^2} + \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}} = 0 \qquad \dots (8.7)$$

$$y = x\frac{dy}{dx} + \frac{1}{\frac{dy}{dx}} \qquad \dots (8.8)$$

Differential equations are classified according to their orders and degrees. The order of a differential equation is the order of the highest derivative or differential occurring in the equation. Thus equations (8.1), (8.3), (8.4), (8.6) and (8.8) are of the *first order*, while equations (8.2), (8.5) and (8.7) are of *second order*. The *degree* of a differential equation is the exponent of the derivative (or the differential) of the highest order in the equation after the equation is freed from negative and fractional powers of the derivatives. Thus the equations (8.1), (8.2), (8.3) and (8.4) are of the *first degree* and (8.5), (8.6), (8.7) and (8.8) are of the *second degree*. Another important classification of differential equations is according to whether they are *linear* or *non-linear*. A differential equation is said to be *linear* if the unknown function and all of its derivatives occurring in the equation, occur only in the first degree.

The differential equations (8.1), (8.2), (8.3) and (8.4) are linear, while (8.5), (8.6), (8.7) and (8.8) are non-linear. In general, an *n*th order linear differential equation is of the form

$$a_0(x)\frac{d^ny}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \ldots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)\,y = \phi(x).$$

The functions $a_0(x), a_1(x), \ldots, a_n(x)$ are called the coefficients of the differential equation. Note that a linear differential equation is always of the first degree but every differential equation of the first degree need not be linear. For example, the equation

$$\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 + y^2 = 0$$

is not linear, though its degree is 1.

EXERCISE 8.1

Determine the order and degree of each of the following equations. State also if they are linear or non-linear.

1. $(xy^2 + x)\,dx + (y - x^2y)\,dy = 0$
2. $\sqrt{1-y^2}\,dx + y\sqrt{1-x^2}\,dy = 0$
3. $t^2\dfrac{d^2s}{dt^2} - st\dfrac{ds}{dt} = s$
4. $\left(\dfrac{ds}{dt}\right)^4 + 3s\dfrac{d^2s}{dt^2} = 0$
5. $y = px + \sqrt{a^2p^2 + b^2}$, where $p = \dfrac{dy}{dx}$
6. $xy\dfrac{dy}{dx} = \left(\dfrac{1+y^2}{1+x^2}\right)(1 + x + x^2)$

7. $x\dfrac{dy}{dx} + \dfrac{3}{\dfrac{dy}{dx}} = y^2$

8. $\dfrac{1}{x}\dfrac{d^2y}{dx^2} = e^x$

9. $\dfrac{d^2y}{dx^2} = \cos 3x + \sin 3x$

10. $\left(\dfrac{ds^2}{dt^2}\right)^2 + 3\left(\dfrac{ds}{dt}\right)^3 + 4 = 0$

8.3 Formation of a Differential Equation

Consider the family of all straight lines passing through the origin (see Fig. 8.1).

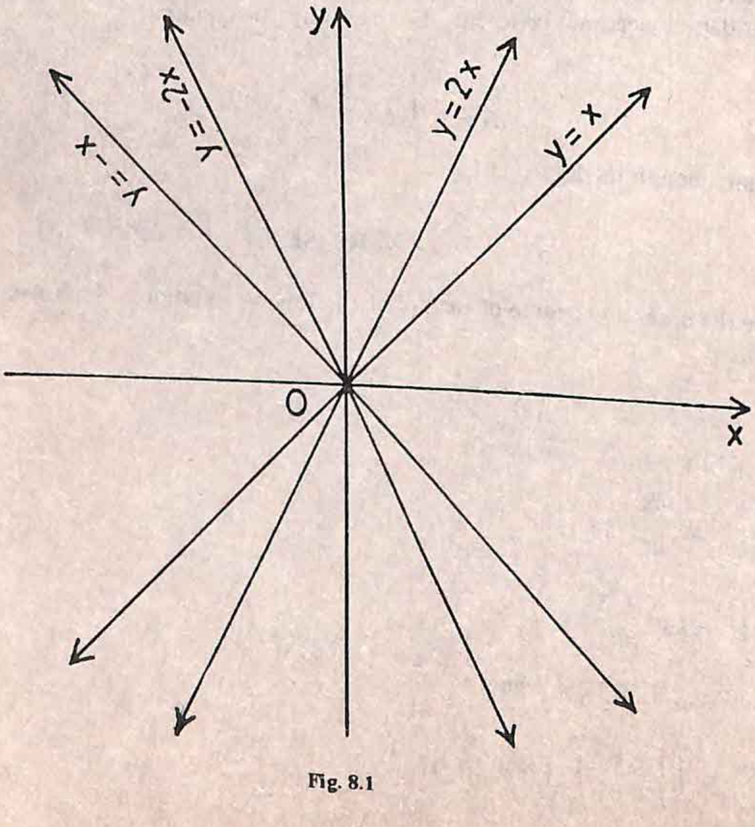


Fig. 8.1

This family of curves can be represented by

$$y = mx \qquad ...(8.9)$$

For different values of m, we get different members of the family. Let us form a differential equation which represents this family of curves. Differentiating the relation (8.9) with respect to x, we get

$$\frac{dy}{dx} = m.$$

Substituting the value of m in (8.9) we get

$$y = x\frac{dy}{dx} \qquad ...(8.10)$$

(8.10) is then the required differential equation.

In general, a family of curves depending on one constant can be represented by

$$f(x, y, a) = 0 \qquad ...(8.11)$$

Differentiating (8.11) with respect to x, we get another relation involving x, y, y' and a, say

$$\phi(x, y, y', a) = 0 \qquad ...(8.12)$$

Eliminating a from the equations (8.11) and (8.12), we get an equation involving x, y, y'. This is the required differential equation of the family of curves represented by (8.11).

Example 8.1

Form the differential equation of the family of curves represented by

$$c\,(y + c)^2 = x^3 \qquad ...(8.13)$$

Solution

Differentiating (8.13) with respect to x, we get

$$2c\,(y + c)\frac{dy}{dx} = 3x^2 \qquad ...(8.14)$$

or

$$\frac{2x^3}{y + c}\,y' = 3x^2$$

or

$$y + c = \frac{2}{3}\,xy'.$$

Hence, $c = \dfrac{2}{3}\,xy' - y$

Substituting the value of c in (8.14), we get

$$\left(\frac{2}{3}\,xy' - y\right)\cdot\frac{2}{3}\,xy'\cdot y' = 3x^2$$

or
$$4xy'^2\,(2xy' - 3y) = 27\,x^2$$

or
$$8x^2\,y'^3 - 12xyy'^2 = 27x^2.$$

Thus, the required differential equation is

$$12y\left(\frac{dy}{dx}\right)^2 = x\left[8\left(\frac{dy}{dx}\right)^3 - 27\right]$$

A family of curves depending on two constants can be represented by a relation of the form

$$f(x, y, a, b) = 0 \qquad \ldots (8.15)$$

A single differentiation of (8.15) with respect to x will yield a relation of the form

$$g(x, y, y', a, b) = 0 \qquad \ldots (8.16)$$

If we again differentiate (8.16) with respect to x, we get a relation of the form

$$h(x, y, y', y'', a, b) = 0 \qquad \ldots (8.17)$$

Eliminating a, b from the equations (8.15), (8.16) and (8.17), we get a relation connecting x, y, y' and y'', say

$$F(x, y, y', y'') = 0$$

This, then, is the required differential equation of the family of curves given by (8.15).

Example 8.2

Form the differential equation representing the family of curves $y = a\cos(x + b)$, where a, b are arbitrary constants.

Solution

Given

$$y = a\cos(x + b) \qquad \ldots (8.18)$$

Differentiating (8.18) with respect to x, we get

$$y' = -\,a\sin(x + b) \qquad \ldots (8.19)$$

Differentiating (8.19) with respect to x, we get

$$y'' = -\,a\cos(x + b) \qquad \ldots (8.20)$$

Eliminating a, b from the equations (8.18), (8.19) and (8.20), we get

$$\frac{d^2y}{dx^2} + y = 0 \qquad \ldots (8.21)$$

which is the required differential equation.

Example 8.3

Find the differential equation of all circles which pass through the origin and whose centres are on the x-axis.

Solution

The general equation of the circles with the desired properties is given by

$$(x-a)^2+y^2=a^2 \qquad \ldots(8.22)$$

We get different members of the family for different values of a. The above equation, upon differentiation with respect to x, yields

$$2(x-a)+2yy'=0$$

or

$$a=x+yy' \qquad \ldots(8.23)$$

Eliminating a from (8.22) and (8.23), we get the required equation. Hence, the required differential equation is

$$\left(y\frac{dy}{dx}\right)^2+y^2=\left(x+y\frac{dy}{dx}\right)^2$$

or

$$y^2=x^2+2xy\frac{dy}{dx} \qquad \ldots(8.24)$$

Remark

We have seen that if an equation between two unknowns contains one arbitrary constant, then the result of eliminating that constant is a differential equation of the first order, and if it contains two arbitrary constants, the result is a differential equation of the second order. Our argument is general, so that if an equation contains n arbitrary constants, the resulting equation, after eliminating the constants, will be a differential equation of the nth order.

Differential equations also arise in connection with many problems in Physics, Chemistry, Biology, Economics, Engineering, etc. We shall illustrate this by considering some examples.

Example 8.4

A population grows at the rate of 5% per year. If $x=x(t)$ denotes the number of individuals in the population after t years, then the rate of change of x is equal to 5% of x. Written in the equation form, we have

$$\frac{dx}{dt}=\frac{5}{100}x$$

which is a differential equation.

Example 8.5

A savings account pays 8% interest per year, compounded continuously. In addition, the income from another investment is credited to the account continuously at the rate of Rs 400 per year. Form the differential equation to model this account.

Solution

Let $x = x(t)$ denote the amount in rupees in the account after t years. Then

$$\frac{dx}{dt} = \frac{8}{100} x + 400$$

which is the required differential equation.

Example 8.6

Assume that a spherical rain drop evaporates at a rate proportional to its surface area. Form a differential equation involving the rate of change of the radius of the rain drop.

Solution

Let $r(t)$ denote the radius (in mm) of the rain drop after t minutes. Since the radius is decreasing as t increases, the rate of change of r must be negative.

If V denotes the volume of the rain drop and S its surface area, we have

$$V = \frac{4}{3} \pi r^3$$

and $$S = 4\pi r^2.$$

It is also given that

$$\frac{dV}{dt} \propto S.$$

Hence,

$$\frac{4}{3} \pi \cdot 3r^2 \frac{dr}{dt} = -k \cdot 4\pi r^2$$

or $$\frac{dr}{dt} = -k.$$

This is, then, the required differential equation.

EXERCISE 8.2

1. Form the differential equation corresponding to

$$y^2 - 2ay + x^2 = a^2$$

by eliminating a.

2. Form the differential equation corresponding to

$$(x-a)^2 + (y-b)^2 = r^2$$

by eliminating a and b.

3. Form the differential equation corresponding to

$$y^2 = m\,(a^2 - x^2)$$

by eliminating m and a.

4. Find the differential equation of all the circles in the first quadrant which touch the coordinate axes.

5. Find the differential equation of all the circles which pass through origin and whose centre lies on y-axis.

6. Find the differential equation of the circles passing through the origin and being their centres on the x-axis.

7. Show that the differential equation that represents all parabolas each of which has a latus rectum $4a$ and whose axes are parallel to x-axis is $2ay_2 + y_1^3 = 0$.

8. Show that the differential equation that represents all parabolas having their axis of symmetry coincident with the axis of x is $yy_2 + y_1^2 = 0$.

8.4 Solution of a Differential Equation

A solution of a differential equation is any function $y = f(x)$, which, when put into the equation changes it into an identity.

Example 8.7

Consider the equation

$$\frac{dy}{dx} = m.$$

Then, as you can easily verify that $y = mx$ is a solution of the equation. Further, $y = mx + c$, for any constant c, is also a solution of the equation.

Example 8.8

Consider the equation

$$\frac{d^2y}{dx^2} + y = 0.$$

It can be easily verified that the functions $y = \sin x$, $y = \cos x$ and, in general, $y = c_1 \sin x + c_2 \cos x$ are solutions of the given equation for any choice of constants c_1 and c_2. The solution which contains a number of arbitrary constants equal to the order of the equation is called the *general solution* or the *complete integral* of the differential equation. Solutions obtained from the general solution, by giving particular values to the constants are called *particular solutions*.

EXERCISE 8.3

In each of the following verify that the given function is a solution of the differential equation:

1. $(y - x)\, dy - (y^2 - x^2)\, dx = 0,\ y = -(1 + x)$
2. $y = x \dfrac{dy}{dx} + a \dfrac{dx}{dy},\ \ y = cx + \dfrac{a}{c}$
3. $y' + y = 0,\ y = ce^{-x}$
4. $xy' - 4y = 0,\ \ y = cx^4$
5. $y''' = 6,\ y = x^3 + ax^2 + bx + c$
6. $y'' + 4y = 0,\ \ y = A \cos 2x + B \sin 2x$

8.5 Equations with Variables Separable

In the previous section, we discussed the notion of a solution of a differential equation. In the rest of the chapter, we shall learn a few techniques of solving differential equations. We shall concern ourselves mainly with differential equations of first order. In this section, we discuss the method of variables separable.

Consider the differential equation

$$\frac{dy}{dx} = f(x, y) \qquad \text{... (8.25)}$$

Suppose the function $f(x, y)$ is of the form

$$f(x, y) = p(x)\, q(y).$$

Then, the equation (8.25) can be written as

$$\frac{dy}{q(y)} = p(x)\,dx \qquad \dots (8.26)$$

In equation (8.26), all the expressions involving y are on one side and those involving x are on the other side. We say that the variables have been separated. Integrating both sides of (8.26), we get

$$\int \frac{dy}{q(y)} = \int p(x)\,dx + c$$

which gives the solution of (8.25).

There is no need to add arbitrary constants of integration on both sides, since they can be combined to give just one arbitrary constant.

Example 8.9

Solve

$$\frac{dy}{dx} = \frac{x^2}{1+y^2}.$$

Solution

The equation can be written as

$$(1+y^2)\,dy = x^2\,dx.$$

Integrating both sides, we get

$$y + \frac{y^3}{3} = \frac{x^3}{3} + c.$$

The above relation gives the general solution of the equation.

In many applications we are not interested in the general solution of a given differential equation, but only in the particular solution satisfying a given initial condition, say, the condition that at some point x_0, the solution $y(x)$ has the value y_0. This is briefly expressed as $y(x_0) = y_0$. A first order differential equation together with an initial condition is called an *initial value problem*.

Example 8.10

Solve the following initial value problem:

$$\frac{dr}{dt} = -rt, \quad r(0) = r_0.$$

Solution

We have

$$\int \frac{dr}{r} = \int -t\,dt$$

or $$\log \frac{r}{c} = \frac{-t^2}{2} \quad \text{(The constant is written as } \log c\text{)}$$

or $$r = ce^{-\frac{t^2}{2}}$$

which is the general solution.

It is given that for $t = 0, r = r_0$. Substituting these values in the general solution, we get $c = r_0$. Hence, the required solution is

$$r = r_0 e^{\frac{t^2}{2}}.$$

Example 8.11

Experiments show that radium disintegrates at a rate proportional to the amount of radium present at the moment. Its *half life*, that is, the time in which 50% of a given amount will disappear is 1590 years. What per cent will disappear in one year ?

Solution

If $y(t)$ be the amount present at time t, then we get from the given data the differential equation

$$\frac{dy}{dt} = -ky$$

Let $y(0) = y_0$. Solving this differential equation, we get

$$\log \frac{y(t)}{c} = -kt$$

Hence, $$y(t) = ce^{-kt}.$$

Using the initial condition $y(0) = y_0$, we get $c = y_0$. Thus, the solution is

$$y(t) = y_0 e^{-kt}.$$

Also, for $t = 1590$, $y(t) = \frac{1}{2} y_0$.

Hence, $$\frac{1}{2} y_0 = y_0 e^{-1590k}$$

or $$k = \frac{\log 2}{1590} \simeq .0004$$

Hence $$y(t) = y_0 e^{-.0004t}$$

$$\therefore \quad y(t+1) = y_0 e^{-.0004(t+1)}$$

$$= y(t)\, e^{-.0004}$$

$$\simeq y(t)(1 - .0004)$$

Since $y(t+1)$ is the amount present after one year, the amount which disappeared in one year $y(t) - y(t+1) \simeq 0.0004\, y(t)$. Hence, % amount disappearing in one year

$$\simeq \frac{0.0004\, y(t)}{y(t)} \times 100 = 0.04\%.$$

Therefore, 0.04% amount will disappear in one year.

Example 8.12

The marginal cost of manufacturing a certain item is given by

$$c'(x) = \frac{dc}{dx} = 2 + 0.15x.$$

Find the total cost function $c(x)$, given that $c(0) = 100$.

Solution

$$\frac{dc}{dx} = 2 + 0.15x$$

or $$\int dc = \int (2 + 0.15x)\, dx + A.$$

Hence, $$c(x) = 2x + 0.075x^2 + A.$$

Given $c(0) = 100$. Hence, A = 100.

$\therefore \quad c(x) = 0.075x^2 + 2x + 100.$

Note

Economists often call the derivative of a function the *marginal value* of the function. Thus, the marginal cost is the derivative of the cost function $c(x)$ where $c(x)$ is the cost of manufacturing x units of a certain item.

EXERCISE 8.4

1. Find the general solution of each of the following equations:

   (i) $\dfrac{dy}{dx} = x^5 + x^2 - \dfrac{2}{x}$

   (ii) $(1 + \cos x)\, dy = (1 - \cos x)\, dx$

   (iii) $\dfrac{dy}{dx} = \cos^3 x \sin^4 x + x\sqrt{2x + 1}$

   (iv) $(x + 2)\dfrac{dy}{dx} = x^2 + 4x - 9$

   (v) $(e^x + e^{-x})\dfrac{dy}{dx} = e^x - e^{-x}$

   (vi) $(x^2 - yx^2)\, dy + (y^2 + x^2y^2)\, dx = 0$

   (vii) $\sec^2 x \tan y\, dx + \sec^2 y \tan x\, dy = 0$

   (viii) $(x - 1)\, y' = 2x^3 y$

   (ix) $y' = (1 + x)\,(1 + y^2)$

   (x) $x \log x\, dy - y\, dx = 0$

   (xi) $\dfrac{dy}{dx} + \sqrt{\dfrac{1 - y^2}{1 - x^2}} = 0$

   (xii) $\left(y - x\dfrac{dy}{dx}\right) = a\left(y^2 + \dfrac{dy}{dx}\right)$

   (xiii) $3e^x \tan y\, dx + (1 - e^x) \sec^2 y\, dy = 0$

   (xiv) $\dfrac{dy}{dx} + 2x = e^{3x}$

   (xv) $\sin^3 x \dfrac{dx}{dy} = \sin y$

2. Solve the following initial value problems:

   (i) $y' = y \tan 2x,\ y(0) = 2$

   (ii) $2xy' = 3y,\ y(1) = 4$

(iii) $y' = 2e^x y^3, \quad y(0) = \dfrac{1}{2}$

(iv) $y' = \sec y, \quad y(0) = 0$

(v) $xyy' = y + 2, \quad y(2) = 0$

8.6 Homogeneous Equations

An equation of the form

$$\frac{dy}{dx} = f(x, y)$$

is said to be *homogeneous* if the function f does not depend on x and y separately, but only on their ratio $\dfrac{y}{x}$ or $\dfrac{x}{y}$. Thus, homogeneous equations are of the form

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right) \qquad \ldots (8.27)$$

Consider the following examples:

$$\frac{dy}{dx} = \frac{y^3 + 3xy^2}{x^3} = \left(\frac{y}{x}\right)^3 + 3\left(\frac{y}{x}\right)^2 \qquad \ldots (8.28)$$

$$\frac{dy}{dx} = \frac{x^2}{y^2} \sin \frac{x}{y} + \frac{y^2}{x^2} \cos \frac{x}{y}$$

$$= \frac{\sin\left(\dfrac{1}{\dfrac{y}{x}}\right)}{\dfrac{y^2}{x^2}} + \frac{y^2}{x^2} \cos\left(\frac{1}{\dfrac{y}{x}}\right) \qquad \ldots (8.29)$$

$$\frac{dy}{dx} = \frac{y^3 + xy}{x^2} \qquad \ldots (8.30)$$

The equations (8.28) and (8.29) are homogeneous, as the right hand side of each can be expressed as a function of $\dfrac{y}{x}$. Since (8.30) cannot be so written, it is not homogeneous.

To solve an equation of the form (8.27), we put $y = vx$. Then

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Substituting in (8.27), we get

$$v + x\frac{dv}{dx} = F(v)$$

or
$$x\frac{dv}{dx} = F(v) - v$$

or
$$\frac{dv}{F(v) - v} = \frac{dx}{x}$$

This equation falls under the 'variables separable' category and can be easily solved.

Example 8.13

Solve

$$(x^2 + 3xy + y^2)\,dx - x^2\,dy = 0.$$

Solution

The equation can be written as

$$\frac{dy}{dx} = \frac{x^2 + 3xy + y^2}{x^2} = 1 + 3\left(\frac{y}{x}\right) + \left(\frac{y}{x}\right)^2.$$

Put $y = vx$. Then

$$\frac{dy}{dx} = v + x\frac{dv}{dx}.$$

Hence,

$$v + x\frac{dv}{dx} = 1 + 3v + v^2$$

or
$$\frac{dv}{(1 + v)^2} = \frac{dx}{x}.$$

Integrating, we get

$$-\frac{1}{1+v} + c = \log x.$$

Substituting the value of v, we get

$$\frac{x}{x+y} + \log x = c$$

which is the required solution.

EXERCISE 8.5

Show that the following equations are homogeneous and solve them :

1. $xy' = x + y$
2. $x^2y' = x^2 + xy + y^2$
3. $(3xy + y^2)\,dx + (x^2 + xy)\,dy = 0$
4. $(x - y)\dfrac{dy}{dx} = x + 3y$
5. $2xyy' = x^2 + 3y^2$

8.7 First Order Linear Equations

A first order linear equation is of the form

$$\frac{dy}{dx} + Py = Q \qquad \ldots(8.31)$$

where P and Q are functions of x or constants.

To solve (8.31), we multiply both sides of the equation by $e^{\int P dx}$ to get

$$e^{\int P dx}\frac{dy}{dx} + Py\, e^{\int P dx} = Qe^{\int P dx} \qquad \ldots(8.32)$$

Now, L.H.S. of (8.32) equals $\dfrac{d}{dx}(y\;e^{\int P dx})$. Hence, we get

$$\frac{d}{dx}(ye^{\int P dx}) = Q\,e^{\int P dx}$$

Integrating, we get

$$y\, e^{\int P dx} = \int Q e^{\int P dx}\, dx + c$$

$$\text{or } y = e^{-\int P dx}\left[\int Q e^{\int P dx}\, dx + c\right]$$

This gives the general solution of (8.31). The expression $e^{\int P dx}$ is called the *integrating factor* of the equation (8.31).

Example 8.14

Solve

$$\frac{dy}{dx} + \frac{y}{x} = e^{-x}$$

Solution

Integrating factor = $e^{\int P dx} = e^{\int \frac{1}{x}\, dx} = x$

Multiplying both sides of the equation by x

we get

$$x\frac{dy}{dx} + y = xe^{-x}$$

or

$$\frac{d}{dx}(xy) = xe^{-x}$$

Integrating, we get

$$xy = -(x+1)\, e^{-x} + c$$

or

$$y = -\left(\frac{x+1}{x}\right) e^{-x} + \frac{c}{x}$$

which is the required solution.

Example 8.15

Solve

$$\frac{dy}{dx} + 2y = \sin x \qquad \ldots (8.34)$$

Solution

Integrating factor = $e^{\int 2dx} = e^{2x}$. Multiplying both sides of the given equation by e^{2x}, we get

$$e^{2x}\frac{dy}{dx} + 2e^{2x}y = e^{2x}\sin x \qquad \dots (8.35)$$

L.H.S. of (8.35) = $\frac{d}{dx}(ye^{2x})$.

Integral of R.H.S. of (8.35)

$$\begin{aligned} I &= \int e^{2x}\sin x\,dx \\ &= \frac{1}{2}e^{2x}\sin x - \frac{1}{2}\int e^{2x}\cos x\,dx \\ &= \frac{1}{2}e^{2x}\sin x - \frac{1}{4}e^{2x}\cos x - \frac{1}{4}\int e^{2x}\sin x\,dx \\ &= \frac{1}{2}e^{2x}\sin x - \frac{1}{4}e^{2x}\cos x - \frac{1}{4}I. \end{aligned}$$

Hence,

$$\begin{aligned} I &= \frac{4}{5}e^{2x}\left[\frac{1}{2}\sin x - \frac{1}{4}\cos x\right] \\ &= \frac{1}{5}e^{2x}\,[2\sin x - \cos x]. \end{aligned}$$

Hence, upon integrating (8.35), we get

$$ye^{2x} = \frac{1}{5}e^{2x}\,[2\sin x - \cos x] + c$$

or

$$y = \frac{1}{5}[2\sin x - \cos x] + ce^{-2x}$$

which is the general solution of the given equation (8.34)

EXERCISE 8.6

1. Find the general solution of the following differential equations:

(i) $y' + 2y = 4x$

(ii) $y' + ay = e^{mx}$

(iii) $y' + y = \cos x$

(iv) $y' + 2y = 6e^{x}$

(v) $y' - 2y = \cos 3x$

(vi) $y' + y = \sin x$

(vii) $x\dfrac{dy}{dx} - 3y = -2kx$

(viii) $x\dfrac{dy}{dx} - y = (x-1)e^{x}$

(ix) $\dfrac{dy}{dx} + y = \cos x - \sin x$

(x) $\dfrac{dy}{dx} + y\tan x = 2x + x^2\tan x$

(xi) $\dfrac{dy}{dx} + \dfrac{y}{x} = x^3$

(xii) $\dfrac{dy}{dx} + \dfrac{y}{x} = \cos x + \dfrac{\sin x}{x}$

(xiii) $y\,dx + (x - y^3)\,dy = 0$

(xiv) $x\dfrac{dy}{dx} + y = x\log x$

(xv) $\dfrac{dy}{dx} + \dfrac{1}{x\log x}y = \dfrac{1}{x}$

(xvi) $y\,dx - (x + 2y^2)\,dy = 0.$

2. The temperature T of a cooling object drops at a rate proportional to the difference $T - S$, where S is the constant temperature of the surrounding medium.

Thus

$$\frac{dT}{dt} = -k(T - S),$$

where $k\ (>0)$ is a constant and t is the time. Solve the differential equation if it is given that $T(0) = 150$.

3. The equation of electromotive forces for an electric circuit containing resistance and self-inductance is

$$E = Ri + L\,\frac{di}{dt},$$

where E is the electromotive force given to the circuit, R, the resistance and L, the coefficient of induction. Find the current i at time t when (i) $E = 0$ and (ii) $E =$ a non-zero constant.

4. Assume that a spherical rain drop evaporates at a rate proportional to its surface area. If its radius originally is 3mm and 1 hour later has been reduced to 2 mm, find an expression for the radius of the rain drop at any time.

5. The line normal to a given curve at each point (x, y) on the curve passes through the point $(2, 0)$. If the curve contains the point $(2, 3)$, find its equation.

6. Experiments show that the rate of inversion of cane sugar in dilute solution is proportional to the concentration $y(t)$ of the unaltered sugar. Suppose that the concentration is $\frac{1}{100}$ at $t = 0$ and is $\frac{1}{300}$ at $t = 10$ hours. Find $y(t)$.

7. A wet porous substance in the open air loses its moisture at a rate proportional to the moisture content. If a sheet hung in the wind loses half its moisture during the first hour, when will it have lost 99%, weather conditions remaining the same ?

8. A bank pays interest by continuous compounding, that is, by treating the interest rate as the instantaneous rate of change of the principal. Suppose in an account interest accrues at 8% per year, compounded continuously. Calculate the percentage increase in such an account over one year. [Take $e^{.08} \simeq 1.0833$].

9. A population grows at the rate of 5% per year. How long does it take for the population to double ?

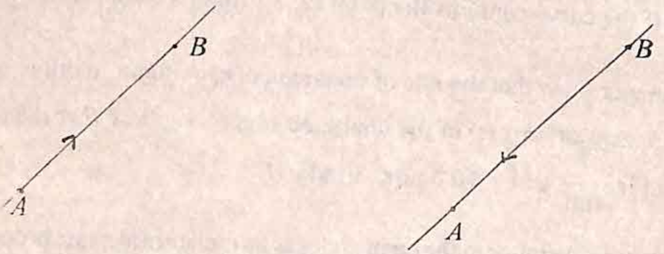
10. Assume that the rise in the price $p = p(t)$ of a product is proportional to the difference between the demand $w(t)$ and the supply $s(t)$ and that the demand depends on the price as a first degree polynomial. Set up a differential equation for the price.

CHAPTER 9

Vector Algebra

9.1 Vectors

Given any straight line AB in a plane or three-dimensional space (3-space), there are two directions that can be associated with the line as shown in Fig. 9.1 by means of arrowheads.



Directed line AB
(i)

Directed line BA
(ii)

Fig. 9.1

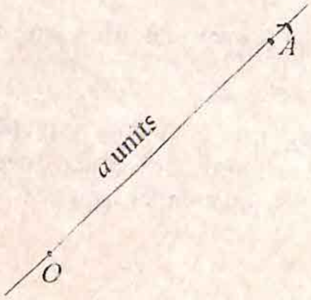


Fig. 9.2

No measurement is involved in these two directions on the line. A line with one of these directions prescribed is called a directed line. If a magnitude in the form of a line segment on this line together with one of the two directions on the line are prescribed we say that we have a *vector* as shown in Fig. 9.2.

It is clear that if the direction and magnitude (a units) have been given, the line segment can be marked on any line in the given direction (pertaining to a class of parallel lines in the plane or 3-space as the case may be). Again, the ends of the line segment may be any two points O and A provided OA has the given length, viz. a units. Thus any *directed line segment* is a vector. If OA is one such vector, O is called its *initial point* and A its *tip* or *end point* or *terminal point* if the direction is from O to A. We employ the symbolism $\overrightarrow{OA}$ or $\vec{a}$ to denote the vector OA in Fig. 9.2. The length of the line segment OA, known as the magnitude of the vector $\overrightarrow{OA}$ is denoted by $|\overrightarrow{OA}|$ or by $|\vec{a}|$. For the purpose of our study it is not necessary to distinguish between two line segments of the same magnitude but lying on two different lines which are parallel to each other provided the directions chosen are one and the same.

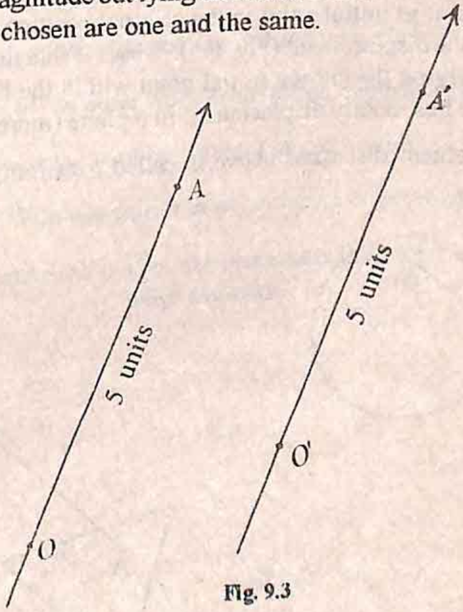


Fig. 9.3

To be explicit, the directed line segments $\overrightarrow{OA}$, $\overrightarrow{O'A'}$ in Fig. 9.3 represent one and the same vector. We say, at times, that they are equal vectors. Note that in the family of directed line segments in space (or a plane) the relation R defined by $\vec{a} R \vec{b}$ if and only if $\vec{a}$ and $\vec{b}$ are directed line segments with the same direction and magnitude (irrespective of initial points) is an equivalence relation. Any member belonging to an equivalence class with reference to this relation represents one and the same vector.

Note

For convenience of printing, vectors are sometimes denoted by letters in thick type known as clarendon type. The overhead arrow symbol is more convenient for hand writing. In figures the direction of a vector is indicated by an arrow head in the middle of the line segment or near the tip or in the extended line segment depending on convenience as in Fig. 9.4.

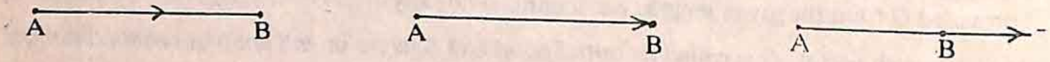
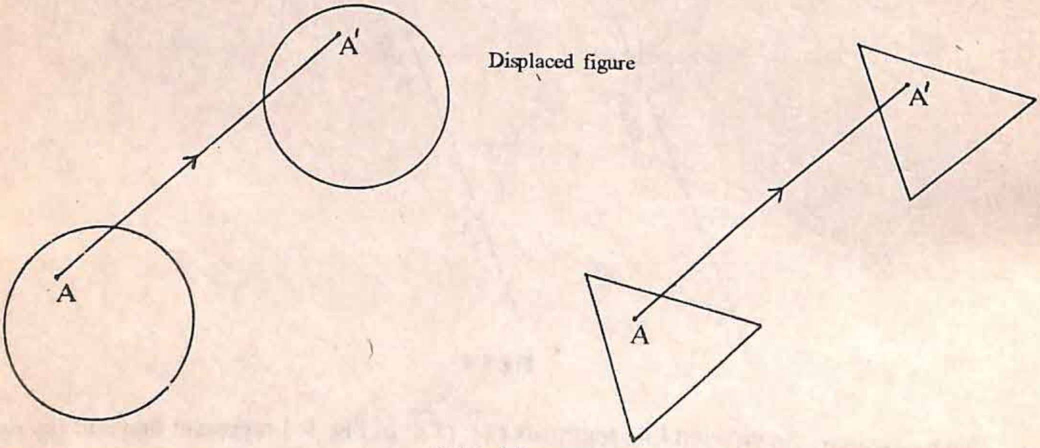


Fig. 9.4

Examples of Vectors

1. If we take a plane figure and displace it along the plane without distorting or rotating the figure, the displacement is completely determined by a vector. The vector is precisely the one with any chosen point as initial point in the original position and the tip as the image of this chosen point by the displacement (Fig. 9.5). Observe that the line segments obtained by this procedure by varying the chosen initial point within the figure are all of the same length and direction. Thus, bodily displacement in a plane (more generally in 3-space) is a vector. The displacement discussed above is called *translation* and $\overrightarrow{AA'}$ is called the *translation-vector*.



Displaced figure

Original figure

Fig. 9.5

2. Any force which is applied to a point or a body is determined by its magnitude and direction and is hence a vector.

3 Velocity, viz. speed together with its direction, is also a vector. The rate of change of velocity for a particle moving with non-constant velocity, known as acceleration, has again both magnitude and direction and is deemed a vector.

Position Vector of a Point

If P is a point in a plane referred to a pair of rectangular axes Ox, Oy, it is clear that the directed line segment OP fixes P uniquely and is called the *position vector* of P. Note that when we speak of position vectors we have in mind a fixed point O which is the original or initial point for the vector.

Scalars

Physicists call an entity having only magnitude but no direction as a *scalar* e.g. **mass**, length, time, temperature. But for the purpose of our study *any real number* (which can also be the magnitude of one of the above entities or a vector) is a *scalar*.

Negative of a Vector

Let $\overrightarrow{OA}$ be a vector. Extend the straight line AO and take A' on AO produced such that $AO = OA'$. Take the line segment OA' together with the direction opposite to that of $\overrightarrow{OA}$ as shown in Fig. 9.6. $\overrightarrow{OA'}$ is defined to be the negative of the vector $\overrightarrow{OA}$. We write $\overrightarrow{OA} = -\overrightarrow{OA}$. If $\overrightarrow{OA}$ is denoted by $\vec{a}$, $\overrightarrow{OA'}$ is denoted by $-\vec{a}$. With our earlier understanding any line segment in the same direction as $\overrightarrow{OA'}$ and of the same length as OA' or OA represents the vector $-\overrightarrow{OA}$. (See Fig. 9.6).

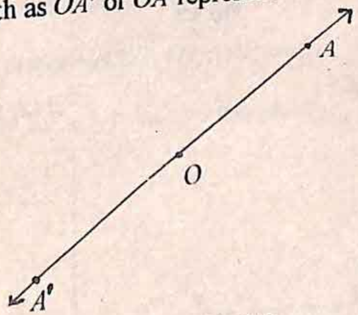


Fig. 9.6

Example 9.1

Let P be a point in a plane referred to a pair of rectangular axes, Ox, Oy. As was pointed out above. $\overrightarrow{OP}$ determines P. What is $-\overrightarrow{OP}$?

Solution

As in Fig. 9.7, if P has coordinates (x, y) with reference to Ox, Oy as axes of coordinates, the construction to represent $-\overrightarrow{OP}$ shows that $(-x, -y)$ will be the coordinates of P' where $\overrightarrow{OP'} = -\overrightarrow{OP}$. In other words, the tip of the negative of a position vector is the reflexion of the tip of the position vector with reference to the origin.

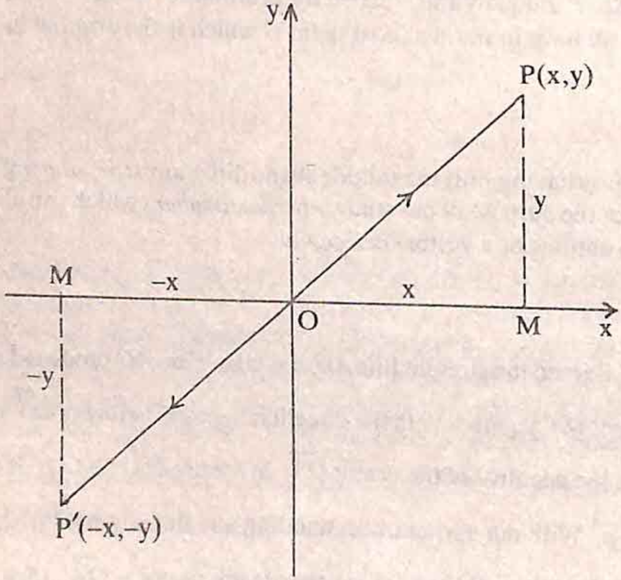


Fig. 9.7

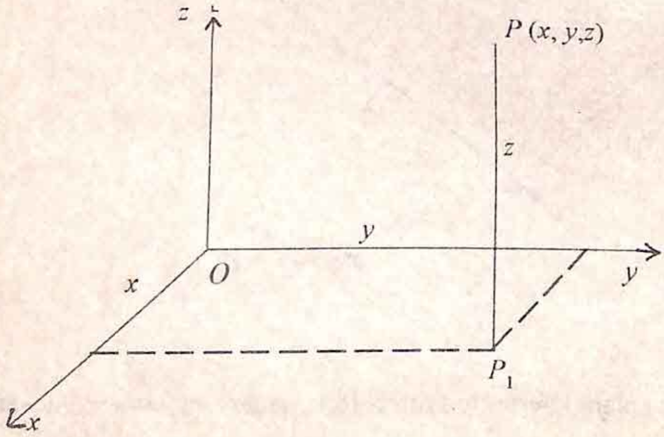


Fig. 9.8

Note

In the three dimensional case too a point P is fixed with reference to a triplet of pairwise orthogonal or perpendicular lines passing through a chosen point O as follows:

If Ox, Oy, Oz are these three lines (Fig. 9.8), consider the foot P_1 of the perpendicular from P onto the plane xOy (which is also called the xy-plane). Observe in this context, that any two intersecting lines or parallel lines fix up a plane in three-dimensional space.

Let the coordinates of P with reference to the rectangular axes Ox, Oy in the plane determined by them be (x, y). Let PP_1 have length z. If P is above the xOy plane, we say that P has coordinates (x, y, z) with reference to the mutually perpendicular axes Ox, Oy, Oz. We write $P\ (x, y, z)$. If P is below the plane xOy it is said to have coordinates $(x, y, -z)$, where z is the numerical value of length PP_1. An equivalent description of x, y, z is that they are perpendicular distances of P from the planes yOz, zOx, xOy with a negative sign attached to those distances in the directions opposite to Ox, Oy, Oz. It is clear that P is fixed by the ordered triplet (x, y, z) and vice-versa. You will learn more about this when you study three-dimensional coordinate geometry in the next chapter.

In example 9.1 you have seen that if P is (x, y) and if $\overrightarrow{OP'} = -\overrightarrow{OP}$, then P' is $(-x, -y)$. This result can be extended to 3-dimensional space also. Thus, if P is (x, y, z) and $\overrightarrow{OP'} = -\overrightarrow{OP'}$ then P' is $(-x, -y, -z)$.

Components of a Vector

If $P\ (x, y)$ is a point (Fig. 9.9) in a plane referred to a pair of rectangular axes Ox, Oy, then vectors $\overrightarrow{OQ}$, $\overrightarrow{OR}$, where Q, R are respectively the feet of the perpendiculars from P to Ox, Oy, are called the *components* of $\overrightarrow{OP}$ in the directions $\overrightarrow{Ox}$, $\overrightarrow{Oy}$ respectively. It is clear that the magnitudes $|\overrightarrow{OQ}|$, $|\overrightarrow{OR}|$ are respectively $|x|$, $|y|$ if P is the point (x, y).

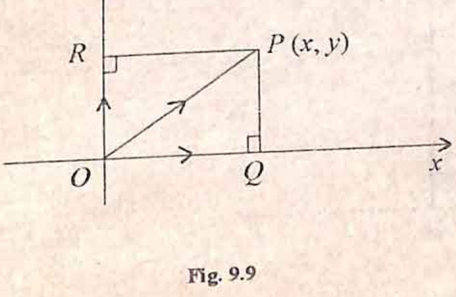


Fig. 9.9

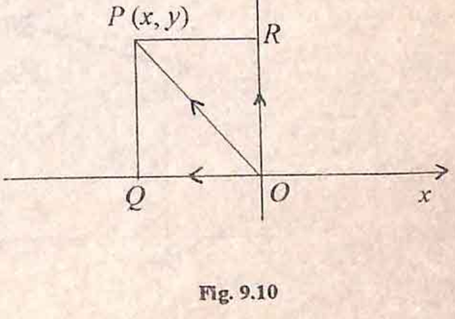


Fig. 9.10

If P were in the second quadrant as in Fig. 9.10 we note that $\overrightarrow{OQ}$ is in a direction opposite to the direction of Ox. In this case x is a negative real number. We can consider similarly the situation of P in the other quadrants and conclude that the magnitude and direction of the components of $\overrightarrow{OP}$ along Ox, Oy are determined by the coordinates of P in magnitude and direction, the direction being opposite to that of Ox, Oy, when x, y are respectively negative.

If P were in three-dimensional space referred to a system of mutually perpendicular axes Ox, Oy, Oz (See Fig. 9.11), it is clear that the lengths OQ, OR, OS where Q, R, S are respectively the feet of the perpendiculars from P to Ox, Oy, Oz are respectively equal to the coordinates x, y, z of P in magnitude as also in direction, the latter in the sense that a negative sign will indicate that OQ, OR, OS are in a direction opposite to that of Ox, Oy, Oz respectively. $\overrightarrow{OQ}$, $\overrightarrow{OR}$, $\overrightarrow{OS}$ are said to be the components of $\overrightarrow{OP}$ along the directions of Ox, Oy, Oz.

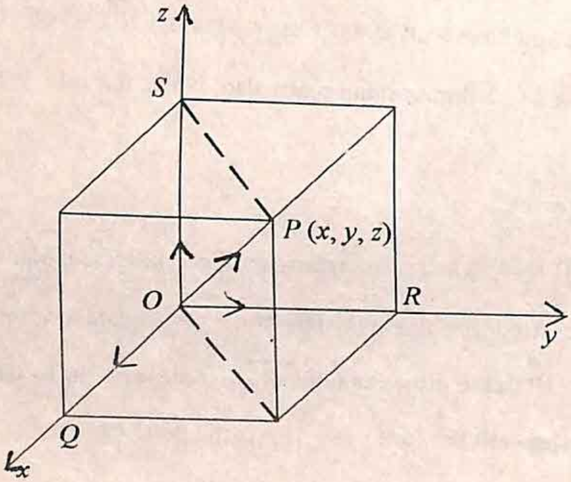


Fig. 9.11

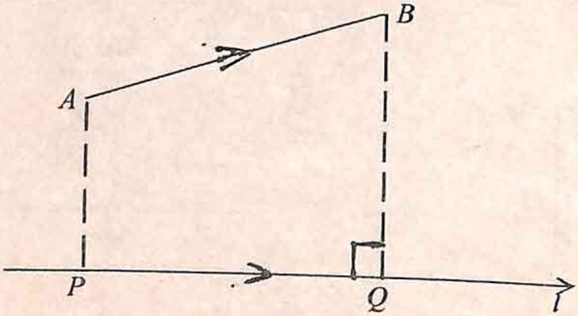
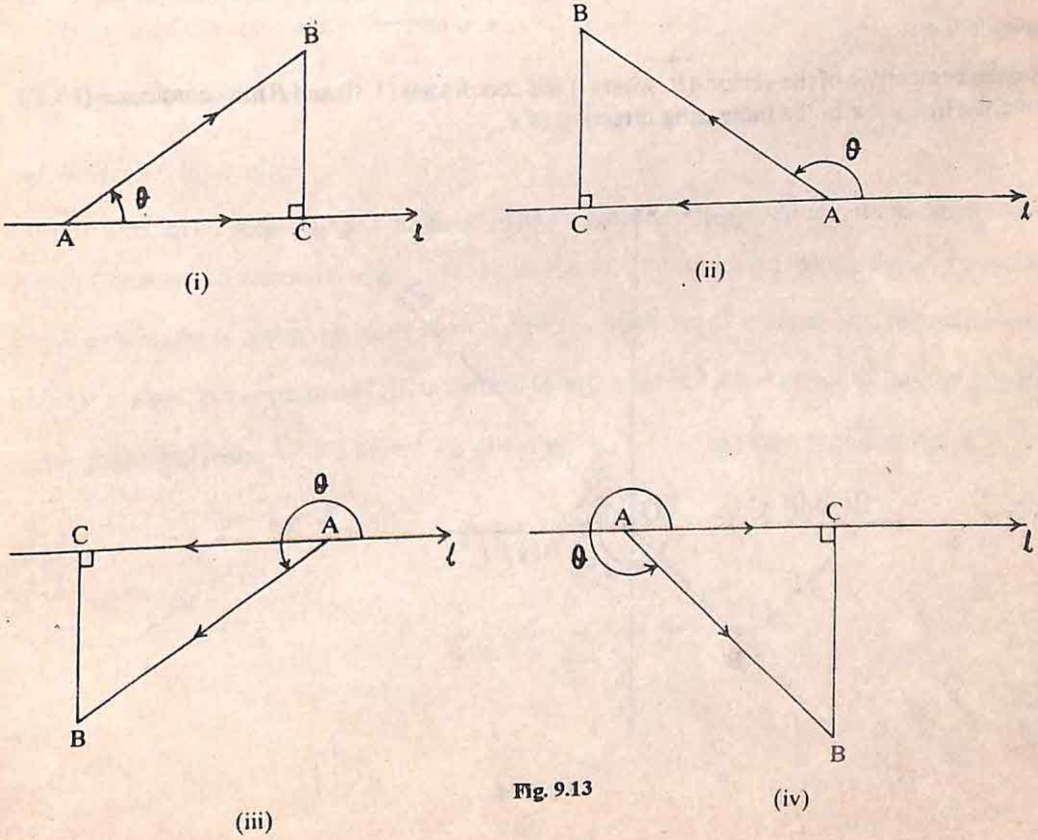


Fig. 9.12

More generally, if $\overrightarrow{AB}$ is any vector in space and l is a directed line the *component* or *projection* of $\overrightarrow{AB}$ on l is defined to be $\overrightarrow{PQ}$ where P, Q are feet of the perpendiculars from A, B respectively to l (See Fig. 9.12).

Remark

It should be remembered that the component or projection of a vector $\overrightarrow{AB}$ in any direction is a vector. Even when the given vector does not meet the directed line on which its component is desired to be found, we can find a line (there are, in fact, many such) which is parallel to the given line and meets the given vector at A. If θ is the magnitude of the angle which the given vector makes with the directed line in the anticlockwise sense, the magnitude of the component is $|\overrightarrow{AB}|\,|\cos\theta|$. Its direction is the given direction of the directed line if $\cos\theta$ is positive and it is the opposite direction if $\cos\theta$ is negative. The figures below illustrate the situation in all the possible cases [See Fig. 9.13(i) - (iv)]. [$\overrightarrow{AC}$ is the component of $\overrightarrow{AB}$ along the direction of l.]



(i) (ii)

(iii) (iv)

Fig. 9.13

Example 9.2

Find the component along Ox of the position vector of the point $P\ (2, 5)$.

Solution

As was observed earlier the absolute value of the x-coordinate of P is the magnitude of the component. Here it is 2. Since the x-coordinate is positive, the direction of the component is along the direction Ox.

Example 9.3

Find the component of the position vector of the point $P\ (-2, -3, 5)$ along the direction Oy of the axes of coordinates.

Solution

The magnitude of the component is $|-3\ | = 3$ units and its direction is along the direction opposite to Oy since the y-coordinate of P is negative.

Example 9.4

Find the component of the vector $\overrightarrow{AB}$, where A has coordinates $(1, 0)$ and B has coordinates $(-3, 0)$ along the line $y = x$ in the increasing direction of x.

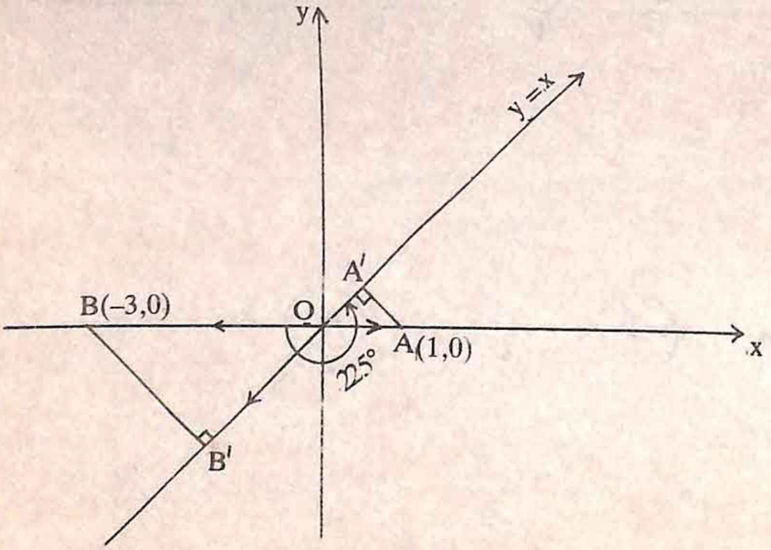


Fig. 9.14

Solution

In this case the angle made by the vector with the directed line whose equation is $y = x$ in the anticlockwise sense is $\theta = 225°$. Hence, the component has magnitude

$$|\overrightarrow{AB}|\ |\cos\theta| = 4\ |\cos 225°| = \frac{4}{\sqrt{2}}$$

Its direction is given by the sign of $\cos\theta$ which is negative and so the component is $\overrightarrow{A'B'}$ in the direction of $y = x$ opposite to that of increasing x as shown in Fig. 9.14.

EXERCISE 9.1

1. Find the component of the vector $\overrightarrow{PQ}$ along the direction Ox if P is (x_1, y_1) and Q is (x_2, y_2) with reference to rectangular coordinate axes Ox, Oy.

2. Find the component of the vector $\overrightarrow{AB}$ where A is $(1, 0)$ and B is $(5, 0)$ along the direction $y = -x$ in the increasing direction of x.

9.2 Operations with Vectors

(a) Scalar Multiplication

Let $\vec{a}$ be a vector corresponding to the directed line segment $\overrightarrow{AB}$ in a plane or in three-dimensional space. Consider the vector $\overrightarrow{AC}$ where $|\overrightarrow{AC}| = |k|\,|\overrightarrow{AB}|$, where $k \neq 0$ is a real number, i.e. a scalar, and the direction is along $\overrightarrow{AB}$ itself or in a direction opposite to it depending respectively on whether k is positive *or* negative. $\overrightarrow{AC}$ is defined to be the vector $k\overrightarrow{AB}$ obtained by multiplying the vector $\overrightarrow{AB}$ by the scalar k. We have $k\overrightarrow{AB}$ given by Fig. 9.15 in the corresponding cases.

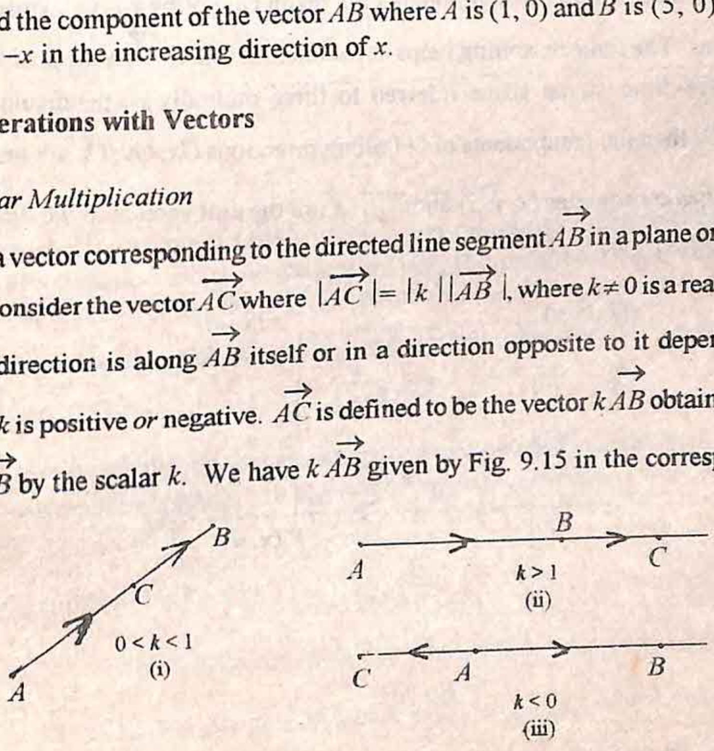


Fig. 9.15

Let, now points of a plane be referred to a pair of rectangular axes Ox, Oy. The vectors in the directions Ox, Oy which have unit magnitude, are denoted by $\vec{i}, \vec{j}$ respectively. These are called *unit vectors* in the directions of the x-axis, y-axis respectively.

With this notation the vector $\overrightarrow{OP}$, where P has coordinates $(x, 0)$, is the vector $x\vec{i}$, x being treated as a scalar. $x\vec{i}$ is in the direction of Ox or in the opposite direction according as $x > 0$ or $x < 0$. Likewise $y\vec{j}$ is the vector $\overrightarrow{OQ}$ of magnitude $|y|$ and has the direction of Oy or the opposite direction according as y is positive or negative. If, now $\overrightarrow{OA}$ is the position vector of the point A (x, y) referred to the axes Ox, Oy and P, Q are the feet of the perpendiculars from A on Ox, Oy respectively (Fig. 9.16), then P has coordinates $(x, 0)$ and Q has coordinates $(0, y)$ so that $\overrightarrow{OP}, \overrightarrow{OQ}$ which are the components of $\overrightarrow{OA}$ along Ox, Oy respectively, are the vectors $x\vec{i}, y\vec{j}$. We observe that the magnitude and direction of the components are taken care of by $x\vec{i}, y\vec{j}$ irrespective of in which quadrant A lies. The same reasoning helps us to conclude that if $\overrightarrow{OA}$ is the position vector of the point A in three-dimensional space referred to three mutually perpendicular axes of coordinates Ox, Oy, Oz then the components of $\overrightarrow{OA}$ along directions Ox, Oy, Oz are respectively $x\vec{i}, y\vec{j}, z\vec{k}$, where A has coordinates (x, y, z) and $\vec{i}, \vec{j}, \vec{k}$ are the unit vectors in the directions of Ox, Oy, Oz.

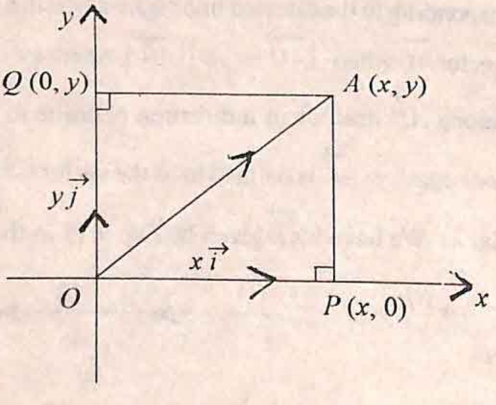


Fig. 9.16

Note: We shall call $\vec{i}, \vec{j}$ or $\vec{i}, \vec{j}, \vec{k}$ all vectors pertaining to basis or basic unit vectors in the plane or 3-space as the case may be.

(b) Addition of Two Vectors

Let $\overrightarrow{AB}$, $\overrightarrow{CD}$ respresent two vectors in a plane or in three-dimensional space (Fig. 9.17). Take the line through A in the direction of $\overrightarrow{CD}$ and cut off a length equal to $|\overrightarrow{CD}|$ from A thus getting the vector $\overrightarrow{AE}$. By our convention it is immaterial whether we consider $\overrightarrow{CD}$ or $\overrightarrow{AE}$. To be explicit they are one and the same vector. Complete the parallelogram whose adjacent sides are AB and AE, calling it $ABFE$. Then $\overrightarrow{AF}$ is defined to be the sum of the vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$. We write

$$\overrightarrow{AB} + \overrightarrow{CD} = \overrightarrow{AB} + \overrightarrow{AE} = \overrightarrow{AF}.$$

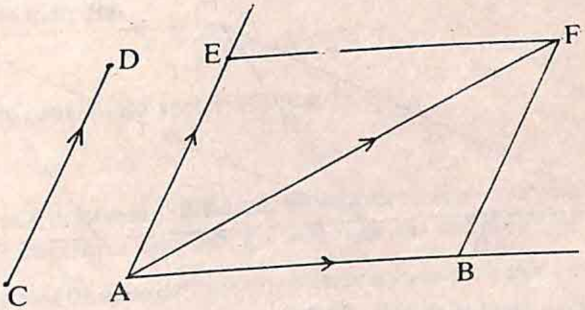


Fig. 9.17

Note that if we replace $\overrightarrow{AB}$ by a vector which represents $\overrightarrow{AB}$ but with initial point C and proceed with the completion of the parallelogram we get a congruent parallelogram with sides parallel to those of $ABFE$. Hence, the diagonal of the new parallelogram will be parallel and equal to that of $ABFE$. In other words, we get another representation for $\overrightarrow{AF}$. More explicitly

$$\overrightarrow{CD} + \overrightarrow{AB} = \overrightarrow{AF} = \overrightarrow{AB} + \overrightarrow{CD}.$$

Note also that both the representations belong to the equivalence class of vectors to which the sum of the two vectors belongs. In algebraic language *addition of two vectors is a commutative operation.*

(c) Difference of Two Vectors

Let, now, $\overrightarrow{AF}$, $\overrightarrow{AB}$ be two vectors with the same initial point A (See Fig. 9.18). Take the vector $\overrightarrow{AB_1} = -\overrightarrow{AB}$. Complete the parallelogram with F, A, B_1 as three vertices. Let the fourth vertex be B_2. Then, by the definition of the sum of two vectors

$$\overrightarrow{AB_2} = \overrightarrow{AF} + \overrightarrow{AB_1}$$
$$= \overrightarrow{AF} + (-\overrightarrow{AB}).$$

We write the vector on the right hand side as

$$\overrightarrow{AF} - \overrightarrow{AB}.$$

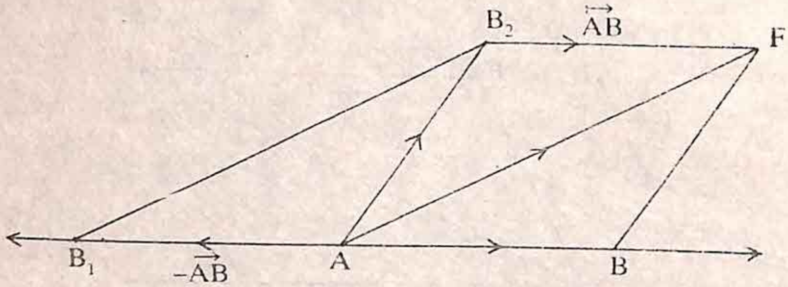


Fig. 9.18

This helps us to define the difference $\vec{a} - \vec{b}$ between two vectors. If $\overrightarrow{AF} = \overrightarrow{AB} + \overrightarrow{CD}$, we can show that

$$\overrightarrow{AF} - \overrightarrow{AB} = \overrightarrow{CD}.$$

This is done once we show that AB_2FB is a parallelogram.

Since $|\overrightarrow{AB}| = |\overrightarrow{AB_1}| = |\overrightarrow{B_2F}|$,

we have $AB = B_2 F$

Also, $AB \,||\, B_2 F$ (by construction)

Hence, $AB_2\, F\, B$ is a parallelogram as desired.

To sum up, we have shown that if $\vec{a} + \vec{b} = \vec{c}$, then

$$\vec{b} = \vec{c} - \vec{a} \qquad \ldots (9.1)$$

$$\vec{a} = \vec{c} - \vec{b} \qquad \ldots (9.2)$$

In other words, we can transpose a summand in a vector sum to the other side with a negative sign as in ordinary addition. The question arises as to what is $\vec{a} - \vec{a}$? We define this to be $\vec{0}$ but assume that this vector has *no direction* though it has magnitude 0. The property mentioned in (9.1) and (9.2) above ensures then that

$$\vec{a} + \vec{0} = \vec{0} + \vec{a} = \vec{a}$$

i.e. $\vec{0}$ is the identity element for vector addition.

Remark

In some books $\vec{0}$ is said to have any direction. Since a vector is an entity with prescribed magnitude and direction, this statement does not go well with the definition of a vector unless this is interpreted to mean that $\vec{0}$ has *no* definite direction or equivalently that its direction is not defined. Development of vector algebra (and analysis) shows that there is no need to have a direction for $\vec{0}$.

The construction of the parallelogram to obtain the sum of two vectors can be dispensed with as follows. If we look at Fig. 9.17,

$$\overrightarrow{CD} = \overrightarrow{AE} = \overrightarrow{BF}.$$

It, therefore, follows that

$$\overrightarrow{AB} + \overrightarrow{CD} = \overrightarrow{AB} + \overrightarrow{BF} = \overrightarrow{AF}.$$

In other words, if $\overrightarrow{AB}$, $\overrightarrow{BF}$ are two vectors with the directions indicated therein, so that they can form the sides of a triangle meeting at B (See Fig. 9.19), then $\overrightarrow{AF}$ is the third side of the triangle which is obtained by joining the points A and F

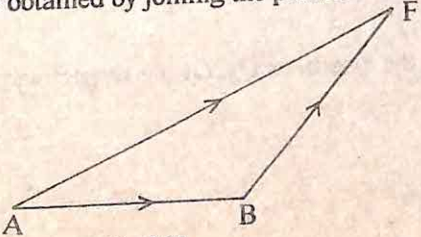


Fig. 9.19

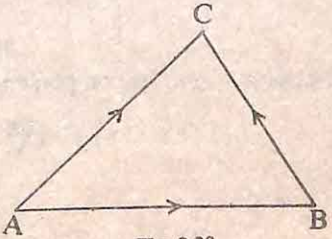


Fig. 9.20

More generally, if ABC is a triangle as in Fig. 9.20 in plane or three-dimensional space (the triangle then lies in a plane), and $\overrightarrow{AB}$, $\overrightarrow{BC}$ are deemed as vectors traced in the anticlockwise direction, their sum is the third side traced in the clockwise direction, viz. $\overrightarrow{AC}$. As any pair of sides of the triangle could be used, it follows that if we have a triangle of vectors, the sum of any two in anticlockwise (clockwise) direction is the third in the clockwise (anticlockwise) direction. Thus, to get the sum of two vectors it suffices to construct the triangle with their tips and the common initial point as vertices. This idea is used in dealing with the resultant force of two forces acting at a point. We shall call this idea the *law of triangle of vectors.*

Note

If $\overrightarrow{AB}$ and $\overrightarrow{BC}$ have the same direction then A, B, C are collinear points and the triangle ABC coincides with the straight line segment ABC. Thus, AC in magnitude is equal to $AB + BC$. The direction of $\overrightarrow{AB} + \overrightarrow{BC}$ is that of either $\overrightarrow{AB}$ or $\overrightarrow{BC}$. So $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$ as shown in Fig. 9.21.

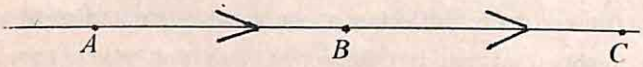


Fig. 9.21

We now look into some special cases of vector addition. Let $\overrightarrow{OP}$ be the position vector of a point P in a plane referred to a pair of perpendicular lines as axes of coordinates Ox, Oy (See Fig. 9.22)

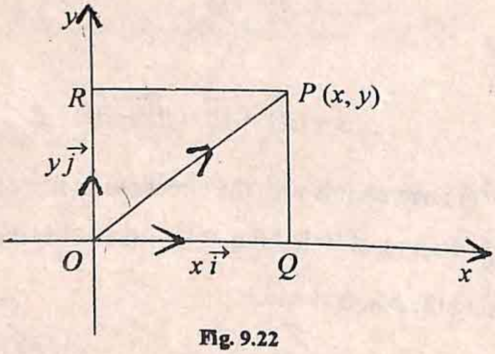


Fig. 9.22

Let P be the point (x, y). Its components in the directions Ox, Oy are respectively

$$x\vec{i} = \overrightarrow{OQ}$$

and

$$y\vec{j} = \overrightarrow{OR}.$$

By the definition of the sum of two vectors, it is clear that

$$\overrightarrow{OQ}+\overrightarrow{OR}=\overrightarrow{OP}$$

or,

$$x\vec{i}+y\vec{j}=\overrightarrow{OP}.$$

In other words, the position vector $\overrightarrow{OP}$ is the sum of its components along the directions of the axes. Conversely, the sum $x\vec{i}+y\vec{j}$ of the vectors $x\vec{i}, y\vec{j}$ along the directions of the axes Ox, Oy respectively, is precisely the position vector of the point $P(x, y)$. Since any two perpendicular lines could have been chosen as axes of coordinates, it follows that if the vector $\overrightarrow{AB}$ has components $\overrightarrow{AC}, \overrightarrow{AD}$ along two perpendicular lines through A, then $\overrightarrow{AB}=\overrightarrow{AC}+\overrightarrow{AD}$.

Note that in the definition of the sum of two vectors the construction of the parallelogram is really in a plane, viz. the plane determined by the intersecting lines AB, AE (Fig. 9.17). Now consider a point $P(x, y, z)$ referred to three mutually perpendicular lines Ox, Oy, Oz in three-dimensional space (Fig. 9.23). It is convenient to assume that the system of axes Ox, Oy, Oz is a right-handed system in the sense that a rotation (in a plane) from Ox to Oy or from Oy to Oz or from Oz to Ox is anticlockwise with reference to Oz, Ox, Oy respectively. Let P_1 be the foot of the perpendicular from P onto the xOy plane (Fig. 9.23). Let Q be the foot of the perpendicular from P_1 onto the axis Ox. Then, by the definition of the coordinates of P, we have

$$\overrightarrow{P_1P}=z\vec{k}, \quad \overrightarrow{QP_1}=y\vec{j}, \overrightarrow{OQ}=x\vec{i}.$$

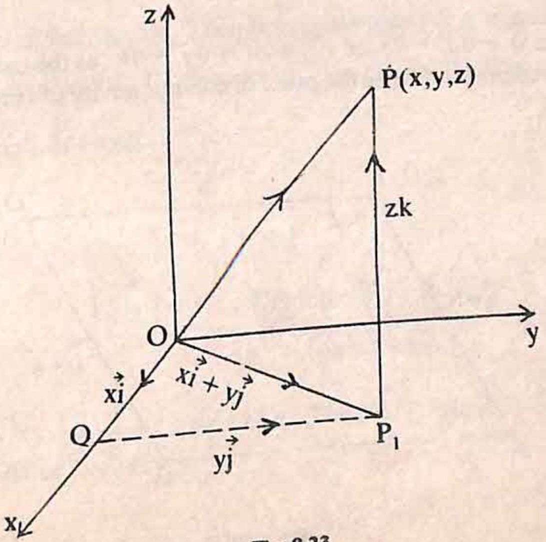


Fig. 9.23

Hence, $\overrightarrow{OP_1} = x\vec{i} + y\vec{j}$. Since $\overrightarrow{P_1P} = z\vec{k}$, it follows that

$$\overrightarrow{OP} = \overrightarrow{OP_1} + \overrightarrow{P_1P} = (x\vec{i} + y\vec{j}\,) + z\vec{k}$$

$$= x\vec{i} + y\vec{j} + z\vec{k}.$$

In the penultimate step there is no need to put brackets since vector addition is seen to be associative as shown below, i.e.

$$(\vec{a} + \vec{b}\,) + \vec{c} = \vec{a} + (\vec{b} + \vec{c}\,)$$

for any three vectors $\vec{a}, \vec{b}, \vec{c}$. This is clear from Fig. 9.24 with the interpretation of sum of two vectors through a triangle of vectors.

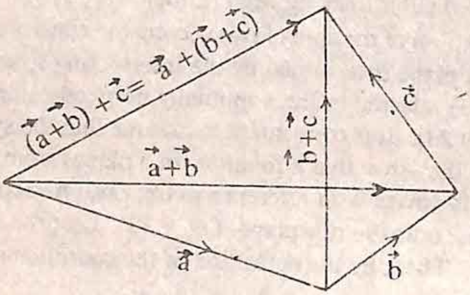


Fig. 9.24

Note

We tacitly assume that $\vec{0} = 0\vec{i} + 0\vec{j}$ or $\vec{0} = o\vec{i} + 0\vec{j} + 0\vec{k}$, as the case may be. By the way, the use of triangle of vectors simplifies the proof of commutativity of vectors as Fig. 9.25 shows.

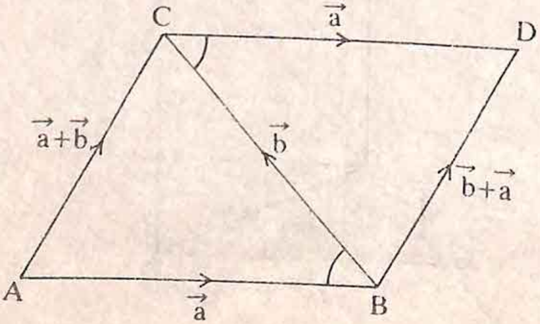


Fig. 9.25

In this figure AB and CD are parallel and $AB = CD$. CD also represents $\vec{a}$ besides $\overrightarrow{AB}$. It turns out that $ABDC$ is a parallelogram and so $AC = BD$, i.e. $|\overrightarrow{AC}| = |\overrightarrow{BD}|$. The directions of $\overrightarrow{AC}$ and $\overrightarrow{BD}$ are also one and the same. These observations are consequences of the fact that AC and BD are parallel to each other. To see this consider the triangles ACB and DBC. In these triangles

$$AB = DC,$$

CB is common and $\angle CBA = \angle DCB \quad (AB \parallel CD)$. So these triangles are congruent and hence

$$\angle ACB = \angle DBC$$

which shows that $AC \parallel BD$.

Distributivity Property (*i*)

Let us now consider addition of scalar multiples of the same vector. Let α, β be scalars, i.e. real numbers and let $\vec{a}$ be a vector. As in Fig. 9.26, let $\overrightarrow{AB}$ be the vector $\vec{a}$.

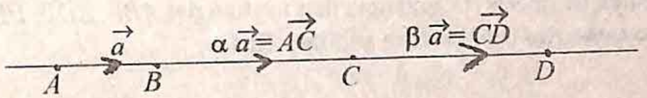

$\vec{a}$
$\alpha\,\vec{a} = \overrightarrow{AC}$
$\beta\,\vec{a} = \overrightarrow{CD}$
A
B
C
D

Fig. 9.26

To begin with let α, β be positive. Then $\alpha\,\vec{a}$ is given by $\overrightarrow{AC}$ which is in the same direction as $\overrightarrow{AB}$ such that $|\overrightarrow{AC}| = \alpha\,|\overrightarrow{AB}|$. To consider $\beta\vec{a}$, we choose the initial point C. If $\overrightarrow{CD} = \beta\vec{a}$, then by the law of triangle of vectors

$$\alpha\vec{a} + \beta\vec{a} = \overrightarrow{AC} + \overrightarrow{CD}$$

$$= \overrightarrow{AD}.$$

Now,

$$|\overrightarrow{AD}| = AC + CD = \alpha\,|\vec{a}| + \beta\,|\vec{a}|$$

$$= (\alpha + \beta)\,|\vec{a}|.$$

Also the direction of $\overrightarrow{AD}$ is the same as that of $\vec{a}$. Thus, $\overrightarrow{AD} = (\alpha + \beta)\,\vec{a}$ by definition of scalar multiple of a vector. Hence, we have

$$(\alpha + \beta)\,\vec{a} = \alpha\,\vec{a} + \beta\,\vec{a}.$$

It is clear that this relation holds even when one or both of α, β are negative since $-\alpha\vec{a} = \alpha(-\vec{a})$ [See §9.2 (a)]

Remark

We call vectors $\vec{a}, \alpha\vec{a}$ which are in the same direction or opposite directions as *collinear* or *parallel vectors.*

Distributivity Property (ii)

Next we consider the scalar multiple of the sum of two vectors. If $\vec{a}, \vec{b}$ are two vectors and α a non-zero scalar (assumed to be positive to begin with) and if $\vec{a}, \vec{b}$ are represented by $\overrightarrow{AB}, \overrightarrow{BC}$ respectively and $\alpha\vec{a}, \alpha\vec{b}$ by $\overrightarrow{AD}, \overrightarrow{DE}$ respectively as in Fig. 9.27, then

$$\frac{AD}{AB} = \frac{|\alpha\vec{b}|}{|\vec{b}|} = \frac{DE}{BC} = \frac{|\alpha\vec{a}|}{|\vec{a}|} = \alpha.$$

Also BC is parallel to DE. The triangles CAB, EAD are, therefore, similar. To see this, let F be the point at which BC meets AE and note that in triangles ABF, ADE, $BF \parallel DE$ and A is a common angle. So these two triangles are similar. Hence,

$$\frac{DE}{BF} = \frac{AD}{AB} = \alpha$$

i.e.,

$$BF = \frac{DE}{\alpha} = BC.$$

So F and C coincide so that AFE is a straight line. Further,

$$\frac{AE}{AF} = \frac{AD}{AB} = \alpha$$

i.e.,

$$AE = \alpha\, AF$$

i.e.,

$$|\overrightarrow{AE}| = \alpha\, |\vec{a} + \vec{b}|.$$

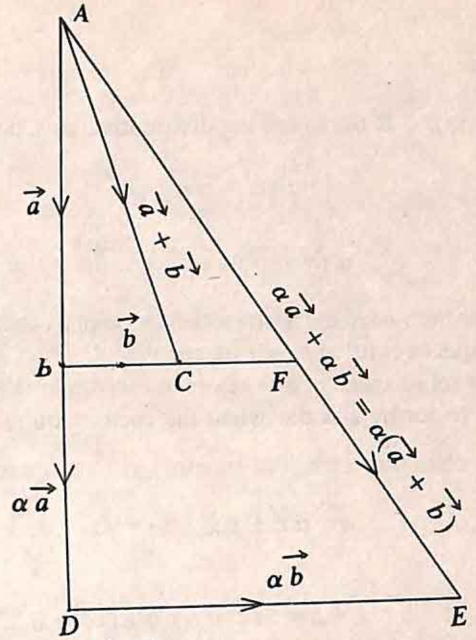


Fig. 9.27

Also, the direction of $\overrightarrow{AE}$ is that of $\vec{a}+\vec{b}$. Thus,

$$\overrightarrow{AE}=\alpha(\vec{a}+\vec{b})=\alpha\vec{a}+\alpha\vec{b}.$$

It is much easier to prove that if α, β are scalars and $\vec{a}$ is a vector, then

$$\alpha(\beta\vec{a})=(\alpha\beta)\vec{a}.$$

This is left as an exercise to the reader.

We shall now summarize the properties of the two operations on the set of all vectors V which we have introduced, viz. scalar multiplication and addition.

(a) V is closed with reference to the binary operation of addition (+) of two vectors.

(b) $\vec{0}$ is the identity element with respect to +.

(c) Every element $\vec{a}\in V$ has an inverse $-\vec{a}$ with reference to addition such that

$$\vec{a}+(-\vec{a})=-\vec{a}+\vec{a}=\vec{0}.$$

(d) Addition of vectors in V is associative.

(e) Addition of vectors in V is commutative.

(f) For each scalar α (in the field of real numbers) and for each $\vec{a}\in V$, $\alpha\vec{a}\in V$.

(g) For every $\vec{a}\in V$, $0\vec{a}=\vec{0}$, $1\vec{a}=\vec{a}$, $\alpha(\beta\vec{a})=(\alpha\beta)\vec{a}$ for any two scalars α, β.

(h) For every $\vec{a}, \vec{b} \in V$, $\alpha, \beta \in \mathbb{R}$ the following distributive laws hold:

$$(\alpha + \beta)\,\vec{a} = \alpha\,\vec{a} + \beta\,\vec{a}$$

$$\alpha\,(\vec{a} + \vec{b}) = \alpha\,\vec{a} + \alpha\,\vec{b}.$$

These properties have motivated the abstract definition of a *vector space* or *linear space* and a study of such abstract spaces entitled *linear algebra*.

The properties of the set or space of all vectors in two dimensions makes it easy to add two vectors or to multiply a vector by a scalar when the vectors are represented in terms of their projections on a pair of rectangular axes. For example, if $\vec{i}$ $\vec{j}$ are basic unit vectors in a plane and

$$\vec{a} = \alpha\,\vec{i} + \beta\vec{j}, \quad \vec{b} = \gamma\,\vec{i} + \delta\,\vec{j}$$

are two vectors, then

$$\vec{a} + \vec{b} = \vec{b} + \vec{a} = (\alpha\,\vec{i} + \beta\vec{j}) + (\gamma\,\vec{i} + \delta\,\vec{j})$$

$$= \alpha\,\vec{i} + \gamma\,\vec{i} + \beta\vec{j} + \delta\,\vec{j}$$

$$= (\alpha + \gamma)\,\vec{i} + (\beta + \delta)\,\vec{j}.$$

This means adding $\vec{a}$, $\vec{b}$ component-wise. Again, if η is a scalar (i.e. a real number),

$$\eta\,(\alpha\,\vec{i} + \beta\vec{j}) = \eta\alpha\,\vec{i} + \eta\beta\vec{j}.$$

Component-wise addition and scalar multiplication holds for 3-dimensional vectors too. If

$$\vec{a} = \alpha\,\vec{i} + \beta\vec{j} + \gamma\,\vec{k}, \quad \vec{b} = \alpha'\,\vec{i} + \beta'\,\vec{j} + \gamma'\,\vec{k},$$

then

$$\vec{a} + \vec{b} = (\alpha\vec{i} + \beta\vec{j} + \gamma\vec{k}) + (\alpha'\vec{i} + \beta'\vec{j} + \gamma'\vec{k})$$

$$= (\alpha + \alpha')\vec{i} + (\beta + \beta')\,\vec{j} + (\gamma + \gamma')\,\vec{k}$$

and

$$\eta\,\vec{a} = \eta\,(\alpha\,\vec{i} + \beta\vec{j} + \gamma\,\vec{k}) = \eta\alpha\,\vec{i} + \eta\beta\vec{j} + \eta\gamma\,\vec{k}$$

Definition

Two vectors $\vec{a}, \vec{b} \neq \vec{0}$ in 2 or 3-dimensional space are said to be *linearly dependent* if and only if their exist non-zero scalars α, β such that $\alpha\,\vec{a} + \beta\vec{b} = \vec{0}$. $\alpha\vec{a} + \beta\,\vec{b}$ is called a linear combination of $\vec{a}$ and $\vec{b}$.

If they are *not* linearly dependent, they are said to be *linearly independent*, i.e. $\alpha\,\vec{a} + \beta\vec{b} = \vec{0} \Rightarrow \alpha = \beta = 0$ if $\vec{a}, \vec{b}$ are linearly independent.

Note that the form $\vec{a} = 2\,\vec{i} + 3\,\vec{j}$ is a linear combination of the vectors $\vec{i}, \vec{j}$. If $\vec{b} = 4\,\vec{i} + 6\,\vec{j}$, then $2\vec{a} - \vec{b} = \vec{0}$, so that $\vec{a}, \vec{b}$ are linearly dependent. In fact, $\vec{b} = 2\,\vec{a}$ so that, according to an earlier definition, $\vec{a}, \vec{b}$ are collinear vectors. This is more generally true in 2 or 3-dimensional space as the following result shows.

Some Important Results

(i) *If $\vec{a}, \vec{b}$ are two non-collinear vectors, and $\alpha\,\vec{a} + \beta\,\vec{b} = \vec{0}$, then $\alpha, \beta = 0$.*

Proof

If $\alpha\,\vec{a} + \beta\,\vec{b} = \vec{0}$, then $\alpha\vec{a} = -\beta\,\vec{b}$. If $\alpha \neq 0$, then $\vec{a} = -\dfrac{\beta}{\alpha}\,\vec{b}$ which shows that the vector $\vec{a}$ is in the direction of $\vec{b}$ if $\left(-\dfrac{\beta}{\alpha}\right)$ is greater than 0 or in the direction opposite to that of $\vec{b}$ if $\left(-\dfrac{\beta}{\alpha}\right)$ is less than 0. If $\beta = 0$, then $\alpha\,\vec{a} = \vec{0}$ which shows that $\alpha = 0$. Thus, if $\alpha, \beta \neq 0$, then $\vec{a}, \vec{b}$ are collinear being in the same direction or in opposite directions. This is contradictory to the hypothesis. Thus, α, β are both 0.

We have also the result:

(ii) *Given two non-collinear non-zero vectors, every vector in the plane determined by them can be expressed as a linear combination of these two vectors.*

Proof

If $\vec{a}, \vec{b}$ are two non-collinear non-zero vectors, we can assume without loss of generality that both $\vec{a}$ and $\vec{b}$ have the same initial point, say O. Let their end points be A, B respectively (Fig. 9.28). Any vector $\vec{x}$ in the plane determined by $\vec{a}, \vec{b}$ can, again, be assumed to have O as initial point. If X were the other terminal point of the vector $\vec{x}$, from X draw lines parallel to OA, OB meeting OB, OA respectively at B_1, A_1. Then, by the definition of the sum of two vectors

$$\overrightarrow{OA}_1 + \overrightarrow{OB}_1 = \vec{x}$$

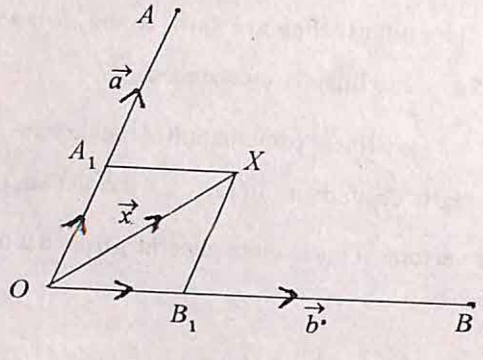


Fig 9.28

Now $\overrightarrow{OA_1}$ and $\overrightarrow{OA}$, $\overrightarrow{OB_1}$ and $\overrightarrow{OB}$ are pairs of vectors in the same direction. Hence, there exist real numbers α and β such that

$$\overrightarrow{OA_1} = \alpha\overrightarrow{OA}, \qquad \overrightarrow{OB_1} = \beta\overrightarrow{OB}.$$

Thus,

$$\vec{x} = \alpha\overrightarrow{OA} + \beta\overrightarrow{OB} = \alpha\vec{a} + \beta\vec{b}$$

and $\vec{x}$ has been expressed as a linear combination of $\vec{a}$ and $\vec{b}$. Thus any non-zero vector in a plane can be expressed as a linear combination of two non-collinear vectors of the same plane.

Note that α, β may also be negative. Fig. 9.29 and Fig. 9.30 illustrate how α or β can be negative. Draw for yourself figures in which both α and β are negative. Trivially,

$$\vec{a} = 1\vec{a} + 0\vec{b}$$
$$\vec{b} = 0\vec{a} + 1\vec{b}$$

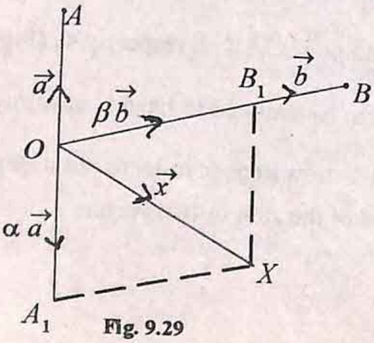


Fig. 9.29

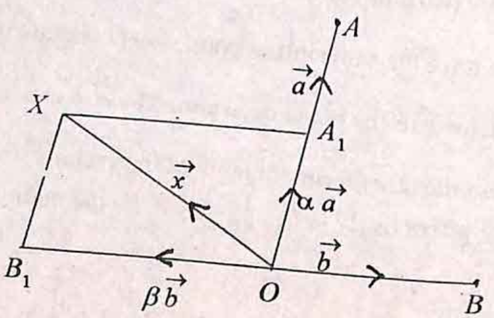


Fig. 9.30

The meaning of the result mentioned as (ii) above is that given any two non-collinear non-zero vectors $\vec{a}, \vec{b}$, any other vector in their plane, say $\vec{c}$, is of the form

$$\vec{c} = \alpha\,\vec{a} + \beta\,\vec{b} \qquad (*)$$

for some α and β. If we now define three vectors $\vec{a}, \vec{b}, \vec{c}$ in 3-dimensional space not all $\vec{0}$, no two of which are collinear, to form a *linearly dependent* system if there exist scalars α, β, γ such that

$$\alpha\,\vec{a} + \beta\,\vec{b} + \gamma\,\vec{c} = \vec{0},$$

then (*) shows that

$$\vec{c} - \alpha\,\vec{a} - \beta\,\vec{b} = \vec{0}$$

or that any three such vectors in a plane are *linearly dependent*. We can define any three vectors to be *linearly independent* if they are not *linearly dependent* e.g. $\vec{i}, \vec{j}, \vec{k}$ the unit vectors in three mutually perpendicular directions are linearly independent. In fact, three vectors $\vec{a}, \vec{b}, \vec{c}$ are linearly dependent if and only if they lie in a plane.

To see this we note that if $\vec{a}, \vec{b}, \vec{c}$ are linearly dependent, there exist α, β, γ *not* all 0 such that

$$\alpha\,\vec{a} + \beta\,\vec{b} + \gamma\,\vec{c} = \vec{0}.$$

So, if $\alpha \neq 0$,

$$\vec{a} = -\frac{\beta}{\alpha}\,\vec{b} - \frac{\gamma}{\alpha}\,\vec{c},$$

i.e. $\vec{a}$ is in the plane determined by $\vec{b}, \vec{c}$.

This can be restated in the form of the following result:

Three vectors in space are linearly independent if and only if they are non-coplanar.

We have also the result:

If $\vec{a}, \vec{b}, \vec{c}$ are three given non-coplanar vectors, then any vector $\vec{v}$ is a linear combination $\alpha\vec{a} + \beta\vec{b} + \gamma\vec{c}$ of $\vec{a}, \vec{b}, \vec{c}$ where α, β, γ are scalars. Also this representation is unique.

Take any point O as origin for vectors. Let

$$\overrightarrow{OA} = \vec{a},\ \overrightarrow{OB} = \vec{b},\ \overrightarrow{OC} = \vec{c},\ \overrightarrow{OP} = \vec{v}$$

The lines OA, OB, OC being not coplanar, determine three different planes

$$BOC, COA, AOB$$

when taken in pairs.

Through P draw planes parallel to the three planes BOC, COA and AOB meeting OA, OB and OC in L, M and N respectively so that we obtain a parallelopiped having OP as a diagonal. We have

$$\vec{v} = \overrightarrow{OP}$$

$$= \overrightarrow{OL} + \overrightarrow{LP}$$

$$= \overrightarrow{OL} + \overrightarrow{LN'} + \overrightarrow{N'P} = \overrightarrow{OL} + \overrightarrow{OM} + \overrightarrow{ON}.$$

There exist scalars α, β and γ such that

$$\overrightarrow{OL} = \alpha\, \overrightarrow{OA} = \alpha\, \vec{a}, \overrightarrow{OM} = \beta\, \overrightarrow{OB} = \beta\, \vec{b}, \overrightarrow{ON} = \gamma\, \overrightarrow{OC} = \gamma\, \vec{c}$$

Thus we have

$$\vec{v} = \alpha\, \vec{a} + \beta\, \vec{b} + \gamma\, \vec{c}.$$

To prove uniqueness let, if possible,

$$\vec{v} = \alpha'\, \vec{a} + \beta' \vec{b} + \gamma'\, \vec{c}.$$

We have, then

$$\alpha\, \vec{a} + \beta\, \vec{b} + \gamma\, \vec{c} = \alpha' \vec{a} + \beta' \vec{b} + \gamma'\, \vec{c}$$

i.e.,

$$(\alpha - \alpha')\, \vec{a} + (\beta - \beta')\, \vec{b} + (\gamma - \gamma')\, \vec{c} = \vec{0}.$$

If $\alpha - \alpha \neq 0$, we have

$$\vec{a} = \frac{\beta - \beta'}{\alpha' - \alpha}\, \vec{b} + \frac{\gamma - \gamma'}{\alpha' - \alpha}\, \vec{c}$$

i.e., $\vec{a}$ is coplanar with $\vec{b}$ and $\vec{c}$ and we arrive at a contradiction to the given hypothesis.

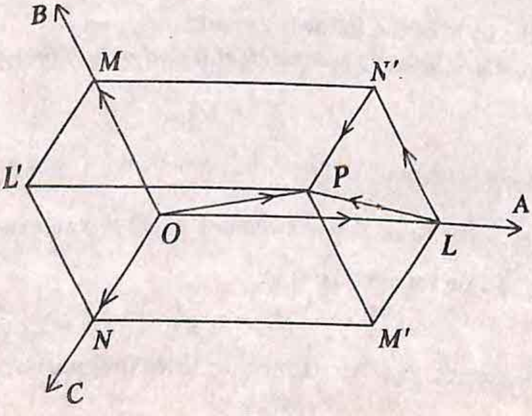


Fig. 9.31

Note: (i) A consequence of the result proved above is that any 4 vectors in space are linearly dependent.

(ii) Three non-coplanar vectors in 3-dimensional space or two non-parallel vectors in 2-dimensional space are said to constitute a *basis* for the space of 3-dimensional or 2-dimensional vectors respectively. $\{\vec{i}, \vec{j}, \vec{k}\}$, $\{\vec{i}, \vec{j}\}$ are basis for the space of 3-dimensional vectors and 2-dimensional vectors respectively. Note that any other vector other than these is expressible as a unique linear combination of $\{\vec{i}, \vec{j}, \vec{k}\}$ or $\{\vec{i}, \vec{j}\}$ as the case may be.

Example 9.5

Find the coordinates of the point which divides the line joining (x_1, y_1, z_1), (x_2, y_2, z_2) in the ratio $m : 1$.

Solution

Let in Fig. 9.31, P be (x_1, y_1, z_1), Q be (x_2, y_2, z_2) and R be (x_3, y_3, z_3) on PQ such that $PR : RQ = m : 1$.

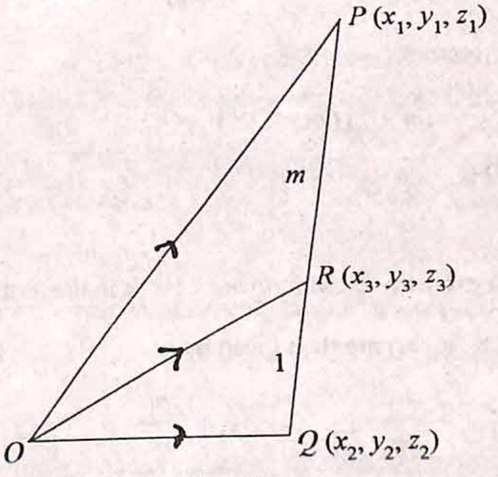


Fig. 9.32

By the definitions of sum of vectors and scalar multiple of a vector,

$$\overrightarrow{RO} + \overrightarrow{OP} = \overrightarrow{RP} = \frac{m}{m+1}\,\overrightarrow{QP} \qquad \ldots (9.3)$$

$$\overrightarrow{QO} + \overrightarrow{OR} = \overrightarrow{QR} = \frac{1}{m+1}\,\overrightarrow{QP} \qquad \ldots (9.4)$$

Multiplying (9.4) by m and subtracting from (9.3),we get

$$\overrightarrow{RO} + \overrightarrow{OP} - m\overrightarrow{QO} - m\overrightarrow{OR} = \vec{0}$$

i.e.,
$$\overrightarrow{OP} + m\overrightarrow{OQ} - (m+1)\,\overrightarrow{OR} = \vec{0}$$

i.e.,
$$\overrightarrow{OP} + m\overrightarrow{OQ} = (m+1)\,\overrightarrow{OR}$$

i.e.,
$$(x_1 + mx_2)\,\vec{i} + (y_1 + my_2)\,\vec{j} + (z_1 + mz_2)\,\vec{k}$$

$$= (m+1)\,x_3\,\vec{i} + (m+1)\,y_3\,\vec{j} + (m+1)\,z_3\,\vec{k}$$

i.e.,
$$x_1 + mx_2 = (m+1)x_3$$
$$y_1 + my_2 = (m+1)y_3$$
$$z_1 + mz_2 = (m+1)z_3$$

i.e.,
$$x_3 = \frac{x_1 + mx_2}{1+m}, y_3 = \frac{y_1 + my_2}{1+m}, z_3 = \frac{z_1 + mz_2}{1+m}$$

Note that, in terms of vectors,

$$(m+1)\,\overrightarrow{OR} = \overrightarrow{OP} + m\overrightarrow{OQ}$$

irrespective of what O is.

Note : Deductions

If the ratio is not given explicitly in the form $m : 1$ but is in the form $m_1 : m_2$ we can set $m = \dfrac{m_1}{m_2}$. The coordinates of $R(x_3, y_3, z_3)$ are then given by

$$x_3 = \frac{x_1 + \dfrac{m_1}{m_2}x_2}{1 + \dfrac{m_1}{m_2}}, y_3 = \frac{y_1 + \dfrac{m_1}{m_2}y_2}{1 + \dfrac{m_1}{m_2}}, z_3 = \frac{z_1 + \dfrac{m_1}{m_2}z_2}{1 + \dfrac{m_1}{m_2}}$$

or what are the same as

$$x_3 = \frac{m_2x_1 + m_1x_2}{m_2 + m_1}, y_3 = \frac{m_2y_1 + m_1y_2}{m_2 + m_1}, z_3 = \frac{m_2z_1 + m_1z_2}{m_2 + m_1}$$

If $R\ (x_3, y_3, z_3)$ is the mid-point of PQ, then the coordinates of R are the means or averages of the corresponding coordinates of P and Q, i.e.

$$x_3 = \frac{x_1 + x_2}{2},\ y_3 = \frac{y_1 + y_2}{2},\ z_3 = \frac{z_1 + z_2}{2}.$$

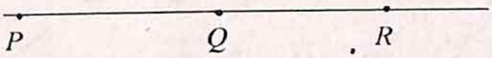


Fig. 9.33

Remark

R may be extended to the segment PQ such that (as in Fig. 9.33) $\frac{PR}{RQ} = m$ (m necessarily greater than 1). In this case R is said to divide PQ externally in the ratio $m : 1$. Given $P(x_1, y_1, z_1)$, $Q(x_2, y_2, z_2)$ we can find the coordinates of R in this case too using vectors. In this case $m > 1$ and $\frac{PR}{RQ} = \frac{m}{1}$ so that $PR = PQ + QR = m\ QR$ or $PQ = (m - 1)\ QR$ or $\frac{PQ}{QR} = \frac{(m-1)}{1}$, i.e. Q divides PR internally in the ratio $(m - 1) : 1$. Thus coordinates of R can be found using this fact.

If R is the mid-point of PQ and O is the origin of vectors, note that

$$\overrightarrow{OR} = \frac{1}{2}\overrightarrow{OP} + \frac{1}{2}\overrightarrow{OQ} = \frac{\overrightarrow{OP} + \overrightarrow{OQ}}{2}$$

The following examples show how vectors could be used to prove some geometrical results.

Example 9.6

Show that the line segment joining the mid-points of two sides of a triangle is parallel to the third side and is half its length.

Solution

Let A of ΔABC be considered as the origin of vectors (Fig. 9.34). Let B_1, C_1 be the mid-points of sides AB, CA respectively. Then

$$\frac{1}{2}\overrightarrow{AB} = \overrightarrow{AB_1},\ \frac{1}{2}\overrightarrow{AC} = \overrightarrow{AC_1}.$$

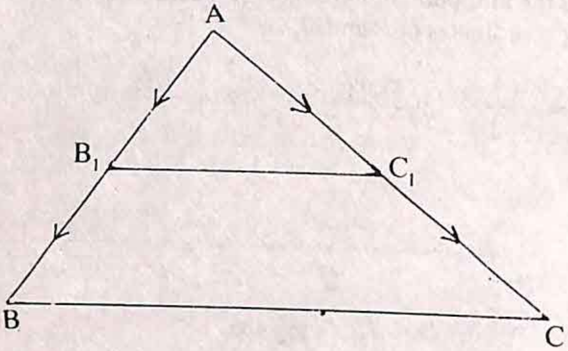


Fig. 9.34

Now, $$\overrightarrow{C_1B_1} = \overrightarrow{C_1A} + \overrightarrow{AB_1} = \overrightarrow{C_1A} - \overrightarrow{B_1A}$$

i.e., $$\begin{aligned}\overrightarrow{C_1B_1} &= \frac{1}{2}\,\overrightarrow{CA} - \frac{1}{2}\,\overrightarrow{BA}\\ &= \frac{1}{2}\,(\overrightarrow{CA} - \overrightarrow{BA})\\ &= \frac{1}{2}\,(\overrightarrow{CA} + \overrightarrow{AB}) = \frac{1}{2}\,\overrightarrow{CB}.\end{aligned}$$

The above equality shows that $\overrightarrow{C_1B_1}$ and $\overrightarrow{CB}$ have the same directions; in other words, $C_1B_1 \,||\, CB$.

Further, $$C_1B_1 = |\overrightarrow{C_1B_1}| = \frac{1}{2}\ |\overrightarrow{BC}| = \frac{1}{2}\ BC.$$

Example 9.7

Show that the diagonals of a quadrilateral bisect each other if and only if it is a parallelogram.

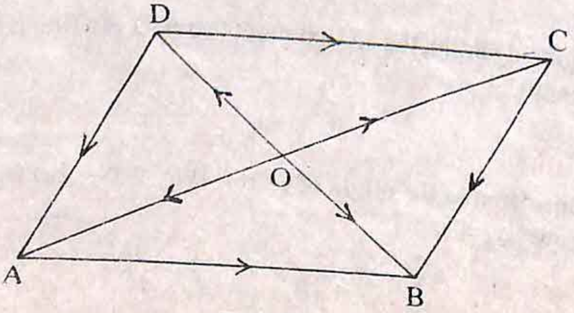


Fig. 9.35

Solution

Let $ABCD$ be the quadrilateral and O be the point of intersection of AC and BD (Fig. 9.34). Choose O as the origin of vectors. O is the mid-point of AC and BD if and only if

$$\overrightarrow{OB} = -\overrightarrow{OD} \quad \text{and} \quad \overrightarrow{OC} = -\overrightarrow{OA}.$$

Equivalently,

$$\overrightarrow{OB} + \overrightarrow{OD} = \overrightarrow{OC} + \overrightarrow{OA} = \overrightarrow{0}$$

or,

$$\overrightarrow{OB} - \overrightarrow{OC} = \overrightarrow{OA} - \overrightarrow{OD}$$

$$\text{i.e.} \quad \overrightarrow{CB} = \overrightarrow{DA}$$

Also, equivalently,

$$\overrightarrow{OB} - \overrightarrow{OA} = \overrightarrow{OC} - \overrightarrow{OD}$$

i.e.,

$$\overrightarrow{AB} = \overrightarrow{DC}.$$

Remark

We have incidentally proved that in a quadrilateral $ABCD$, the diagonals bisect each other if and only if a pair of its opposite sides are parallel and of the same length. Also we have proved that a quadrilateral in which a pair of opposite sides are parallel and of equal length is a parallelogram.

Example 9.8

In Fig. 9.36, $ABCD$ is a parallelogram. E, F are mid-points of BC, CD respectively. AE, AF meet the diagonal BD at Q, P respectively. Show that P, Q trisect DB.

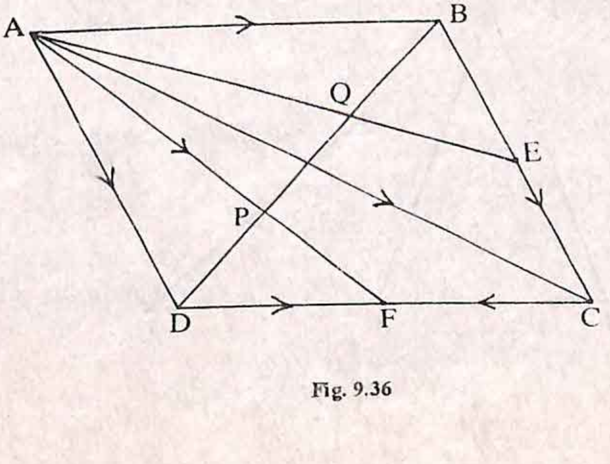


Fig. 9.36

Solution

$$\overrightarrow{AD} + \overrightarrow{DF} = \overrightarrow{AF} = \overrightarrow{AC} + \overrightarrow{CF}.$$

So

$$\begin{aligned} 2\overrightarrow{AF} &= \overrightarrow{AD} + \overrightarrow{DF} + \overrightarrow{AC} + \overrightarrow{CF} \\ &= \overrightarrow{AD} + \overrightarrow{AC} \quad \text{since } \overrightarrow{DF} = -\overrightarrow{CF} \\ &= \overrightarrow{AD} + \overrightarrow{AB} + \overrightarrow{BC} \end{aligned}$$

i.e., $\dfrac{2\overrightarrow{AF} + \overrightarrow{0}}{3} = \dfrac{2\overrightarrow{AD} + \overrightarrow{AB}}{3}$, since $\overrightarrow{AD} = \overrightarrow{BC}$ ($ABCD$ being a parallelogram)

Treating A as the origin of vectors, it follows (see the note to the solution of Example 9.5) that the line segment joining the tips of the position vectors B and D has the point at which it is divided in the ratio 2 : 1 same as the point at which the line segment joining the tips of the position vectors F and A (i.e. origin) divided in the ratio 2 : 1. In other words, $DP = \frac{1}{3} BD$. It is similarly shown (give the details) that

$$DQ : QB = AQ : QE = 2 : 1.$$

Thus, $DP = BQ = \frac{1}{3} BD$ and hence $DP = BQ = PQ$

as required to be shown.

Example 9.9

Prove that the three medians of a triangle meet at a point called the centroid of the triangle which divides each of the medians in the ratio 2 : 1.

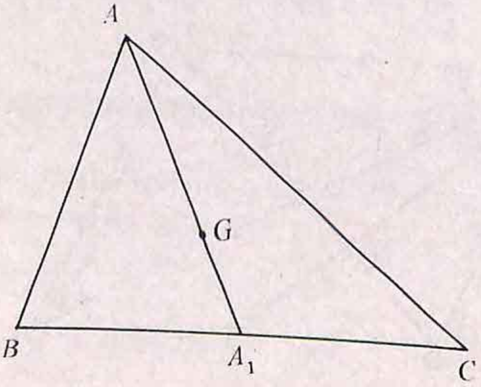


Fig. 9.37

Solution.

Let G be the point on the median through the vertex A of the triangle ABC (Fig. 9.37) such that $AG : GA_1 = 2 : 1$, A_1 being the mid-point of BC. Let O be the origin of vectors. Now

$$\overrightarrow{GA} = \overrightarrow{GO} + \overrightarrow{OA} = \overrightarrow{OA} - \overrightarrow{OG}.$$

Similarly,

$$\overrightarrow{A_1G} = \overrightarrow{OG} - \overrightarrow{OA}_1.$$

Also, A_1 being the mid-point of BC,

$$\overrightarrow{OA}_1 = \frac{1}{2}\ (\overrightarrow{OB} + \overrightarrow{OC}\,) \qquad \text{(See note under Example 9.5).}$$

Hence,

$$\overrightarrow{A_1G} = \overrightarrow{OG} - \frac{1}{2}\ (\overrightarrow{OB} + \overrightarrow{OC}) \qquad \ldots (9.5)$$

But $\overrightarrow{GA} = 2\overrightarrow{A_1G}$ so that

$$\begin{aligned} \overrightarrow{GA} &= \overrightarrow{OA} - \overrightarrow{OG} \\ &= 2\overrightarrow{OG} - (\overrightarrow{OB} + \overrightarrow{OC}\,) \qquad \text{(from 9.5)} \end{aligned}$$

Thus,

$$3\overrightarrow{OG} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}$$

or,

$$\overrightarrow{OG} = \frac{\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}}{3}.$$

If G_1 were the point on the median BB_1 such that

$$BG_1 : G_1B_1 = 2 : 1,$$

the same argument would show that

$$\overrightarrow{OG}_1 = \frac{\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}}{3}.$$

In other words, G and G_1 coincide. By considering the third median we conclude that all the three medians pass through the point G such that

$$\vec{OG} = \frac{\vec{OA} + \vec{OB} + \vec{OC}}{3}$$

and which divides each of the medians in the ratio 2 : 1.

Example 9.10

Find the lengths of the sides of the triangle ABC whose vertices have position vectors $A(3, 4, 5)$, $B(4, 3, 2)$, $C(3, 6, -3)$.

Solution

If O is the origin, from Fig. 9.38,

$$\begin{aligned} \vec{BC} &= \vec{OC} - \vec{OB} \\ &= 3\vec{i} + 6\vec{j} - 3\vec{k} - (4\vec{i} + 3\vec{j} + 2\vec{k}) \\ &= -\vec{i} + 3\vec{j} - 5\vec{k}. \end{aligned}$$

So $|BC|$ = length of the side BC

$$= \sqrt{1^2 + 3^2 + 5^2} = \sqrt{35}.$$

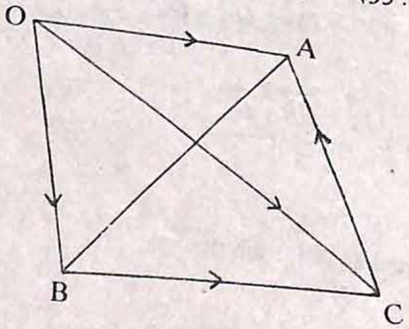


Fig. 9.38

Similarly,

$$\begin{aligned} CA &= |\vec{OA} - \vec{OC}| \\ &= |0\vec{i} - 2\vec{j} + 8\vec{k}| \end{aligned}$$

$$= \sqrt{0^2 + 2^2 + 8^2} = \sqrt{68}$$

and

$$AB = |\overrightarrow{OB} - \overrightarrow{OA}|$$

$$= |\vec{i} - \vec{j} - 3\vec{k}|$$

$$= \sqrt{1^2 + 1^2 + 3^2} = \sqrt{11}.$$

Example 9.11

Show that the three points whose position vectors are $A\ (-2, 3, 5)$, $B\ (1, 2, 3)$, $C\ (7, 0, -1)$ are collinear.

Solution

The three points lie in a plane. In that plane if $AB + BC = AC$ then A, B, C will be collinear points. Now,

$$AB = |\overrightarrow{AB}| = |3\vec{i} - \vec{j} - 2\vec{k}|$$

$$= \sqrt{3^2 + 1^2 + 2^2} = \sqrt{14}$$

$$BC = |\overrightarrow{BC}| = |6\vec{i} - 2\vec{j} - 4\vec{k}|$$

$$= \sqrt{6^2 + 2^2 + 4^2} = \sqrt{56} = 2\sqrt{14}$$

$$CA = |\overrightarrow{CA}| = |-9\vec{i} + 3\vec{j} + 6\vec{k}|$$

$$= \sqrt{9^2 + 3^2 + 6^2} = \sqrt{126} = 3\sqrt{14}.$$

So,

$$AB + BC = CA.$$

Note

It can happen that $BC + CA = AB$ or $CA + AB = BC$ for A, B, C to be collinear.

Example 9.12

Show that the points $A\ (6, -7, 0)$, $B\ (16, -19, -4)$, $C\ (0, 3, -6)$, $D\ (2, -5, 10)$ are such that AB and CD intersect at the point $P\ (1, -1, 2)$.

Solution

$$AP = |\overrightarrow{AP}| = |-5\vec{i} + 6\vec{j} + 2\vec{k}|$$
$$= \sqrt{5^2 + 6^2 + 2^2} = \sqrt{65}$$
$$PB = |\overrightarrow{PB}| = |-15\vec{i} + 18\vec{j} + 6\vec{k}|$$
$$= \sqrt{15^2 + 18^2 + 6^2} = 3\sqrt{65}$$
$$AB = |\overrightarrow{AB}| = |10\vec{i} - 12\vec{j} - 4\vec{k}|$$
$$= \sqrt{10^2 + 12^2 + 4^2} = 2\sqrt{65}.$$

So $PA + AB = PB$. Hence, P, A, B are collinear points. Likewise

$$CP = |\overrightarrow{CP}| = \sqrt{1^2 + 4^2 + 8^2} = 9$$
$$PD = |\overrightarrow{PD}| = \sqrt{1^2 + 4^2 + 8^2} = 9$$
$$CD = |\overrightarrow{CD}| = \sqrt{2^2 + 8^2 + 16^2} = 18.$$

So $CP + PD = CD$. Hence, C, P, D are collinear. Therefore, AB, CD pass through P.

Example 9.13

Write down the equation of the line through the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$.

Solution

The position vectors of P and Q are respectively

$$\overrightarrow{OP} = x_1\vec{i} + y_1\vec{j} + z_1\vec{k}$$
$$\overrightarrow{OQ} = x_2\vec{i} + y_2\vec{j} + z_2\vec{k}.$$

If R is any point with position vector

$$\overrightarrow{OR} = x\vec{i} + y\vec{j} + z\vec{k},$$

then there exists a real number λ such that $PR : RQ = \lambda : 1$. Then

$$x\vec{i} + y\vec{j} + z\vec{k} = \overrightarrow{OR} = \frac{x_1\vec{i} + y_1\vec{j} + z_1\vec{k} + \lambda(x_2\vec{i} + y_2\vec{j} + z_2\vec{k})}{1 + \lambda}$$

Hence,

$$x = \frac{x_1 + \lambda x_2}{1 + \lambda}, \ y = \frac{y_1 + \lambda y_2}{1 + \lambda}, \ z = \frac{z_1 + \lambda z_2}{1 + \lambda}$$

i.e.,

$$x - x_1 = \frac{\lambda}{1 + \lambda} (x_2 - x_1)$$

$$y - y_1 = \frac{\lambda}{1 + \lambda} (y_2 - y_1)$$

$$z - z_1 = \frac{\lambda}{1 + \lambda} (z_2 - z_1).$$

Eliminating λ, we get

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

which is the equation of the line PQ since any $R(x, y, z)$ therein satisfy these equations.

EXERCISE 9.2

1. If $\vec{a}, \vec{b}$ are position vectors of the points $(1, -1)$, $(-2, m)$, find the value of m for which $\vec{a}$ and $\vec{b}$ are collinear.
2. If the position vector $\vec{a}$ of the point $(5, n)$ is such that $|\vec{a}| = 13$, find the value of n.
3. If $A = (0, 1)$, $B = (1, 0)$, $C = (1, 2)$, $D = (2, 1)$, prove that $\overrightarrow{AB} = \overrightarrow{CD}$.
4. Find the coordinates of the tip of the position vector which is equivalent to $\overrightarrow{AB}$, where
   (i) $A = (3, 1)$, $B = (5, 0)$
   (ii) $A = (-1, 3)$, $B = (-2, 1)$.
5. If A, B, C, D are the vertices of a parallelogram and A, B, C have respectively the following coordinates:
   (i) $(2, 3)$; $(1, 4)$, $(0, -2)$
   (ii) $(-2, -1)$, $(3, 0)$, $(1, -2)$,

   find the coordinates of D.

6. $\vec{a}$ is a position vector whose tip is $(1, -3)$. Find the coordinates of the point B such that $\overrightarrow{AB} = \vec{a}$ if A has coordinates $(-1, 5)$.
7. If $x\vec{i} + y\vec{j}$ is a vector referred to two rectangular axes in a plane, show that
$$|x\vec{i} + y\vec{j}| = \sqrt{x^2 + y^2}.$$
Derive a similar result for a vector in 3-dimensional space.
8. Show that any three vectors in the same plane are linearly dependent, i.e. if $\vec{a}, \vec{b}, \vec{c}$ are three vectors, show that there exist α, β, γ not all zero such that
$$\alpha\vec{a} + \beta\vec{b} + \gamma\vec{c} = \vec{0}.$$
What about vectors more than three in number ?
9. Show that in 3-dimensional space, three vectors are linearly independent if and only if they do not lie in a plane.
10. Show that any four vectors in 3-dimensional space are necessarily linearly dependent.
11. If, $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ are any four vectors in 3-dimensional space with the same initial point and such that
$$3\vec{a} - 2\vec{b} + \vec{c} - 2\vec{d} = \vec{0},$$
show that the terminals A, B, C, D of these vectors are coplanar. Find the point at which AC and BD meet. Find the ratio in which P divides AC and BD.
12. Find the distance between the points $A\ (2, 3, 1)$, $B\ (-1, 2, -3)$.
13. If A, B, C have position vectors $(2, 0, 0)$, $(0, 1, 0)$ $(0, 0, 2)$, show that ΔABC is isosceles.

9.3 Products of Two Vectors

Two products are usually defined between two vectors. They are

(a) Scalar or dot product (inner product in the abstract form)

(b) Vector or cross product.

If $\vec{a}, \vec{b}$ are two vectors the product in (a) is denoted by $\vec{a} \cdot \vec{b}$ while the product in (b) is denoted by $\vec{a} \times \vec{b}$. The names dot and cross product are respectively in keeping with the notation. The main difference between the two products is that the dot product is a *scalar* while the cross product is a *vector*, again, as the names indicate.

(a) *Scalar Product or Dot Product of Two Vectors.*

Definition

If $\vec{a} = a_1\vec{i} + a_2\vec{j}, \vec{b} = b_1\vec{i} + b_2\vec{j}$ are two vectors in a plane referred to a pair of rectangular axes along whose positive directions $\vec{i}$, $\vec{j}$ are unit vectors, the scalar product $\vec{a} \cdot \vec{b}$ is defined by

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2.$$

If, on the other hand, $\vec{a}, \vec{b}$ are vectors in 3-dimensional space with $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$, $\vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$ with reference to a right handed system $\vec{i}$, $\vec{j}$, $\vec{k}$ of basic unit vectors in mutually perpendicular directions, then $\vec{a} \cdot \vec{b}$ is defined by

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3.$$

The binary operation of $(\vec{a}, \vec{b}) \to \vec{a} \cdot \vec{b}$ is called scalar multiplication.

Let the position vectors of $\vec{a}, \vec{b}$ make angles θ_1, θ_2 respectively with the positive direction of the x-axis (in case $\vec{a}$, $\vec{b}$ are taken to be vectors in a plane), the angles being measured from Ox in the anticlockwise sense (See Fig. 9.39). Then,

$$\cos \theta_1 = \frac{a_1}{\sqrt{a_1^2 + a_2^2}}, \ \sin \theta_1 = \frac{a_2}{\sqrt{a_1^2 + a_2^2}}$$

$$\cos \theta_2 = \frac{b_1}{\sqrt{b_1^2 + b_2^2}}, \ \sin \theta_2 = \frac{b_2}{\sqrt{b_1^2 + b_2^2}}$$

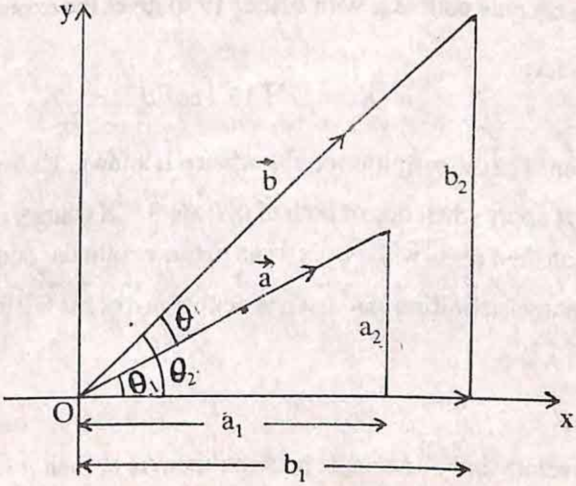


Fig. 9.39

In the preceding expressions, we can write

$$\sqrt{a_1^2 + a_2^2} = |\vec{a}|, \qquad \sqrt{b_1^2 + b_2^2} = |\vec{b}|.$$

Then,

$$\cos(\theta_2 - \theta_1) = \cos\theta_2 \cos\theta_1 + \sin\theta_2 \sin\theta_1$$

$$= \frac{b_1}{|\vec{b}|} \cdot \frac{a_1}{|\vec{a}|} + \frac{b_2}{|\vec{b}|} \cdot \frac{a_2}{|\vec{a}|}$$

$$= \frac{a_1 b_1 + a_2 b_2}{|\vec{a}|\,|\vec{b}|}$$

$$= \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|\,|\vec{b}|}.$$

Thus, if we define $\theta_2 - \theta_1 = \theta$ to be the angle between $\vec{a}$ and $\vec{b}$, then in terms of the scalar product

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|\,|\vec{b}|}. \qquad \text{... (9.6)}$$

If we define the angle between $\vec{a}$ and $\vec{b}$ to be the angle through which $\vec{a}$ is to be rotated in the anticlockwise sense to become collinear with $\vec{b}$, then (9.6) gives the expression for this angle θ.

Again, from (9.6), we have

$$\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}| \cos\theta \qquad \text{... (9.7)}$$

so that (9.7) can be taken to be the definition of $\vec{a} \cdot \vec{b}$ when θ is known. Unfortunately the geometric definition of θ does not apply when one or both of $\vec{a}, \vec{b}$ are $\vec{0}$. Of course, if $\vec{a}, \vec{b}$ are collinear and are in the same direction then $\theta = 0$, while $\theta = \pi$ when they are collinear and in opposite directions. Thus, (9.7) is an alternative definition of $\vec{a} \cdot \vec{b}$ when neither of $\vec{a}$ or $\vec{b}$ is $\vec{0}$. However, in the excluded case we can define $\vec{a} \cdot \vec{b} = 0$.

Alternative Definition

If $\vec{a}, \vec{b}$ are non-zero vectors and if the angle between them is θ, then $\vec{a} \cdot \vec{b}$ is defined by

$$\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}| \cos\theta.$$

If one or both of $\vec{a}$, $\vec{b}$ is $\vec{0}$, then $\vec{a} \cdot \vec{b}$ is defined to be 0.

In the first instance if we have $\vec{a}, \vec{b} \neq \vec{0}$ and $\vec{a}$, $\vec{b}$ are in perpendicular directions then $\theta = \dfrac{\pi}{2}$ so that $\vec{a} \cdot \vec{b} = 0$.

Note that the original definition of $\vec{a} \cdot \vec{b}$ holds even when one or both of $\vec{a}, \vec{b}$ are $\vec{0}$. The alternative definition, however, shows that the value of $\vec{a} \cdot \vec{b}$ does not depend on the choice of the basic vectors $\vec{i}, \vec{j}$.

We adopt the alternative definition in 3-space too though we have reconciled the two definitions in the case of co planar vectors. For, according to our convention, two vectors can be considered to have the same initial point and can, therefore, be assumed to lie in a plane. So we could have referred the two vectors to a pair of rectangular axes in their plane. This justifies the use of the alternative definition in 3-space too, provided it is shown that the change of the system of basic vectors in mutually perpendicular directions does not affect the value of the scalar product. To see this let $\vec{i'}$, $\vec{j'}$, $\vec{k'}$ be basic unit vectors in three mutually perpendicular directions obtained by rotating the system $\vec{i}$, $\vec{j}$, $\vec{k}$ bodily around origin.

Let

$$\vec{i} = i_1 \vec{i'} + i_2 \vec{j'} + i_3 \vec{k'}$$

$$\vec{j} = j_1 \vec{i'} + j_2 \vec{j'} + j_3 \vec{k'}$$

$$\vec{k} = k_1 \vec{i'} + k_2 \vec{j'} + k_3 \vec{k'}$$

Then

$$\begin{aligned} \vec{a} &= a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k} \\ &= (a_1 i_1 + a_2 j_1 + a_3 k_1) \vec{i'} + (a_1 i_2 + a_2 j_2 + a_3 k_2) \vec{j'} \\ &\quad + (a_1 i_3 + a_2 j_3 + a_3 k_3) \vec{k'} \\ \vec{b} &= (b_1 i_1 + b_2 j_1 + b_3 k_1) \vec{i'} + (b_1 i_2 + b_2 j_2 + b_3 k_2) \vec{j'} \\ &\quad + (b_1 i_3 + b_2 j_3 + b_3 k_3) \vec{k'}. \end{aligned}$$

Thus, in terms of the new system $\vec{i'}$, $\vec{j'}$, $\vec{k'}$, by definition we have

$$\vec{a} \cdot \vec{b} = \sum_{r=1}^{3} (a_1 i_r + a_2 j_r + a_3 k_r)(b_1 i_r + b_2 j_r + b_3 k_r)$$

$$\begin{aligned}
&= a_1b_1\,(i_1^2+i_2^2+i_3^2) + a_2b_2\,(j_1^2+j_2^2+j_3^2) \\
&\quad + a_3b_3\,(k_1^2+k_2^2+k_3^2) + (a_1b_2+a_2b_1)\,(i_1j_1+i_2j_2+i_3j_3) \\
&\quad + (a_2b_3+a_3b_2)\,(j_1k_1+j_2k_2+j_3k_3) + (a_3b_1+a_1b_3)\,(k_1i_1+k_2i_2+k_3i_3) \\
&= a_1b_1+a_2b_2+a_3b_3 = \vec{a}\cdot\vec{b}
\end{aligned}$$

as defined for the basic vectors $\vec{i}, \vec{j}, \vec{k}$. In the above simplification we have used the fact that

$$i_1^2+i_2^2+i_3^2 = j_1^2+j_2^2+j_3^2 = k_1^2+k_2^2+k_3^2 = 1$$

$$i_1j_1+i_2j_2+i_3j_3 = j_1k_1+j_2k_2+j_3k_3 = k_1i_1+k_2i_2+k_3i_3 = 0$$

since $\vec{i}', \vec{j}', \vec{k}'$ are unit vectors which are pairwise perpendicular.

If θ were the angle between $\vec{a}$ and $\vec{b}$, by definition the angle between $\vec{b}$ and $\vec{a}$ would be $2\pi-\theta$. From trigonometry, we have

$$\cos(2\pi-\theta) = \cos\theta.$$

So $$\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}.$$

Again, in the context of the scalar product, by the property of the cosine function, we could choose θ to be that angle θ or $2\pi-\theta$ whichever satisfies the relations $0 \le \theta \le \pi$ or what is the same, the smaller of the two angles θ and $2\pi-\theta$, both being non-negative.

Note

$|\vec{b}|\cos\theta$ ($|\vec{a}|\cos\theta$) is the projection of the magnitude of $\vec{b}$ ($\vec{a}$) in the direction of $\vec{a}$ ($\vec{b}$). Also the scalar product $\vec{a}\cdot\vec{b}$ is positive or negative or 0 according as angle θ is acute or obtuse or right angle. This interpretation is relevant in the context of the concept of *work done by a force*.

Definition

If $\vec{F}$ is a force which causes a displacement $\vec{s}$, the *work done* by $\vec{F}$ is the product of the magnitude of the displacement with the magnitude of the component of $\vec{F}$ along the direction of the displacement $\vec{s}$ (Fig. 9.40).

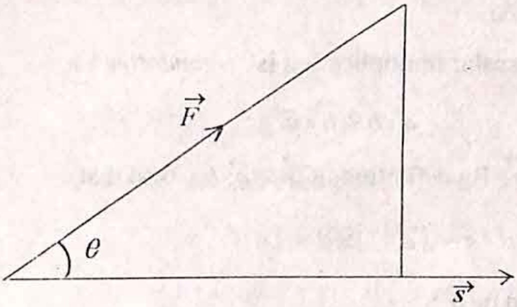


Fig. 9.40

If angle between $\vec{F}$ and $\vec{s}$ is θ, the work W done by $\vec{F}$ is given by $W = |\vec{s}|\,|\vec{F}|\cos\theta = \vec{F}\cdot\vec{s}$.

If $\vec{F}$ be the resultant of two forces $\vec{F}_1$ and $\vec{F}_2$, acting at the same point and causing the same displacement $\vec{s}$, then, we have, $\vec{F} = \vec{F}_1 + \vec{F}_2$ and $\vec{F}\cdot\vec{s} = (\vec{F}_1 + \vec{F}_2)\cdot\vec{s} = \vec{F}_1\cdot\vec{s} + \vec{F}_2\cdot\vec{s}$, which expresses that the work done by the resultant is equal to the sum of the works done by the forces separately.

Example 9.14

A particle acted on by two forces $4\vec{i} + 3\vec{j}$ and $3\vec{i} + 2\vec{j}$ is displaced from the point $\vec{i} + 2\vec{j}$ to $5\vec{i} + 4\vec{j}$. Find the total work done by these forces.

Solution

If $\vec{F}$ be the resultant of two forces, then

$$\vec{F} = (4\vec{i} + 3\vec{j}) + (3\vec{i} + 2\vec{j})$$
$$= 7\vec{i} + 5\vec{j}$$

If A and B are the points $\vec{i} + 2\vec{j}$ and $5\vec{i} + 4\vec{j}$ respectively, then the distance

$$\vec{d} = \overrightarrow{AB} = 4\vec{i} + 2\vec{j}$$

Therefore, the work done by the given forces

= work done by their resultant $\vec{F}$

$$= \vec{F}\cdot\vec{d}$$

$$= (7\vec{i} + 5\vec{j})\cdot(4\vec{i} + 2\vec{j})$$

= 38 units.

Properties of the Scalar Product

We have already seen that scalar multiplication is *commutative* i.e.

$$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$$

for any pair of vectors $\vec{a}$, $\vec{b}$. By definition, if $\vec{b} = \vec{a}$, $\theta = 0$ so that

$$\vec{a} \cdot \vec{a} = |\vec{a}|\ |\vec{a}| = |\vec{a}|^2.$$

Sometimes, $\vec{a} \cdot \vec{a}$ is denoted by $\vec{a}^{\,2}$.

If $\vec{b} = -\vec{a}$ i.e. $\vec{b}$ has the same magnitude as $\vec{a}$ but has opposite direction to $\vec{a}$, then

$$\vec{a} \cdot (-\vec{a}) = |\vec{a}|\ |\vec{a}| \cos \pi = -|\vec{a}|^2.$$

Next, when $\vec{a} \cdot \vec{b} = 0$

i.e. $$\vec{a} \cdot \vec{b} = |\vec{a}|\ |\vec{b}| \cos \theta = 0$$

and $\vec{a}$, $\vec{b} \neq 0$, then this means that $\cos \theta = 0$. In other words

$$\theta = \frac{\pi}{2}$$

since $0 \leq \theta \leq \pi$ as we may assume without loss of generality. This amounts to saying that $\vec{a}$ and $\vec{b}$ are in perpendicular directions. In this case we say that $\vec{a}, \vec{b}$ are orthogonal vectors. It is clear that if $\vec{a}, \vec{b}$ are orthogonal vectors, then $\vec{a} \cdot \vec{b} = 0$. Thus, we have the following result:

$\vec{a}, \vec{b} \neq \vec{0}$, $\vec{a} \cdot \vec{b} = 0$ if and only if $\vec{a}, \vec{b}$ are *orthogonal* vectors.

Of course we have defined $\vec{a} \cdot \vec{b} = 0$ if either $\vec{a}$ or $\vec{b}$ (or both) is $\vec{0}$.

If $\vec{a}, \vec{b}$ are vectors in the same direction then

$$\vec{a} \cdot \vec{b} = |\vec{a}|\ |\vec{b}|$$

since $\theta = 0$ in this case too.

If $\vec{i}$, $\vec{j}$ are unit vectors in a plane in perpendicular directions, then what we have seen above makes us conclude

$$\begin{aligned} \vec{i} \cdot \vec{i} &= \vec{i}^{\,2} = 1 \\ \vec{j} \cdot \vec{j} &= \vec{j}^{\,2} = 1 \\ \vec{i} \cdot \vec{j} &= \vec{j} \cdot \vec{i} = 0 \end{aligned} \qquad \ldots (9.8)$$

If $\vec{i}, \vec{j}, \vec{k}$ are unit vectors in three mutually perpendicular directions in 3-dimensional space, then we have

$$\left.\begin{aligned} \vec{i}^{\,2} = \vec{j}^{\,2} = \vec{k}^{\,2} = 1 \\ \vec{i}\cdot\vec{j} = \vec{j}\cdot\vec{k} = \vec{k}\cdot\vec{i} = 0 \end{aligned}\right\} \quad \ldots (9.9)$$

If, however, $\vec{a}, \vec{b}$ are unit vectors, not necessarily orthogonal, then

$$\vec{a}\cdot\vec{b} = |\vec{a}|\ |\vec{b}|\cos\theta$$

$$= 1\cdot 1\cos\theta = \cos\theta$$

when θ is the angle between $\vec{a}$ and $\vec{b}$.

We know that if $\vec{a}$ is a vector and α is a scalar, then $\alpha\vec{a}$ is a vector. Let $\vec{b}$ be another vector and consider

$$(\alpha\vec{a})\cdot\vec{b}.$$

By definition of scalar product

$$(\alpha\vec{a})\cdot\vec{b} = |\alpha\vec{a}|\ |\vec{b}|\cos\phi$$

when ϕ is the angle between $\alpha\vec{a}$ and $\vec{b}$. But $\vec{a}$ and $\alpha\vec{a}$ have the same direction if $\alpha > 0$. In this case ϕ is equal to the angle θ between $\vec{a}$ and $\vec{b}$. Thus

$$(\alpha\vec{a})\cdot\vec{b} = \alpha(\vec{a}\cdot\vec{b}) \quad \text{if } \alpha > 0.$$

If $\alpha < 0$, then $\alpha\vec{a}$ is in the direction opposite to that of $\vec{a}$ (Fig. 9.41). In this case $\phi = \pi + \theta$ and

$$\cos\phi = \cos(\pi + \theta) = -\cos\theta.$$

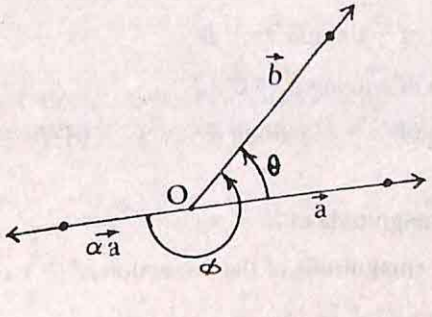


Fig. 9.41

Thus,
$$\begin{aligned}(\alpha\vec{a})\cdot\vec{b} &= |\alpha\vec{a}|\;|\vec{b}|\cos\phi\\ &= |\alpha|\;|\vec{a}|\;|\vec{b}|\cos\phi\\ &= \alpha\,|\vec{a}|\;|\vec{b}|\cos\phi\\ &= \alpha\,(\vec{a}\cdot\vec{b}).\end{aligned}$$

Thus, in either case,

$$(\alpha\vec{a})\cdot\vec{b} = \alpha\,(\vec{a}\cdot\vec{b}).$$

That this is true even when $\alpha = 0$ is trivial.

We have therefore, the following result:

If $\vec{a}, \vec{b}$ are vectors and α is a scalar then

$$(\alpha\vec{a})\cdot\vec{b} = \vec{a}\cdot(\alpha\vec{b}) = \alpha\,(\vec{a}\cdot\vec{b}).$$

More generally, if α, β are two scalars,

$$(\alpha\vec{a})\cdot(\beta\vec{b}) = \alpha\beta\,(\vec{a}\cdot\vec{b}).$$

Note

The more general result in the second part is deducible from the first part by two applications of the latter.

We next show that scalar multiplication of vectors distributes over vector addition. To be more explicit:

If $\vec{a}, \vec{b}, \vec{c}$ are vectors, then

$$\vec{a}\cdot(\vec{b}+\vec{c}) = \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c}.$$

To establish this result, let $\overrightarrow{AB}$, $\overrightarrow{BC}$ stand for $\vec{b}, \vec{c}$ respectively so that $\overrightarrow{AC}$ represents $\vec{b}+\vec{c}$. Let A', B', C' be the projections of A, B, C respectively in the direction of $\vec{a}$.

Magnitude of projection of $\vec{b}$ along $\vec{a} = A'B'$

Magnitude of projection of $\vec{c}$ along $\vec{a} = B'C'$

Magnitude of projection of $(\vec{b}+\vec{c})$ along $\vec{a} = A'C' = (A'B' + B'C$.

Thus,

$$\begin{aligned}\vec{a}\cdot(\vec{b}+\vec{c}) &= (\text{magnitude of } \vec{a})\\ &\quad\times(\text{magnitude of the projection of } (\vec{b}+\vec{c}) \text{ along } \vec{a})\\ &= |\vec{a}|\;(A'B' + B'C')\end{aligned}$$

$$= |\vec{a}| \cdot A'B' + |\vec{a}| \cdot B'C' \qquad \text{(Distributive law for scalars)}$$

$$= |\vec{a}| \cdot \text{Projection of } \vec{b} \text{ along } \vec{a}$$

$$+ |\vec{a}| \cdot \text{Projection of } \vec{c} \text{ along } \vec{a}$$

$$= \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}.$$

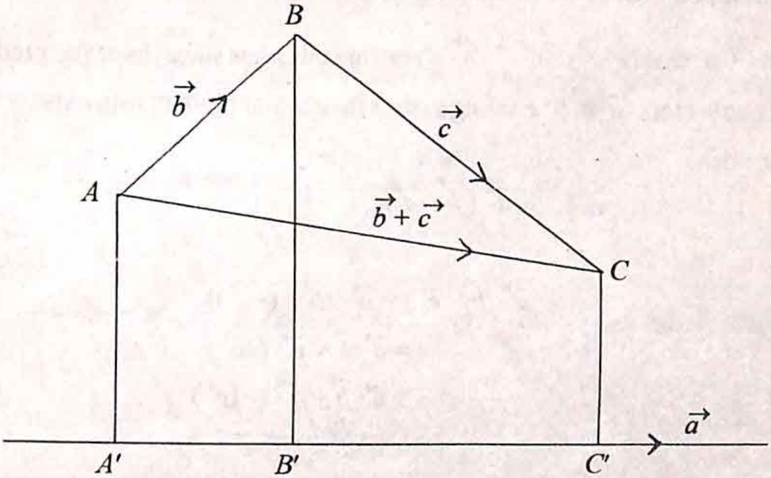


Fig. 9.42

Fig. 9.42 has been drawn in such a way that the vector $\vec{a}$ makes acute angles with both the vectors $\vec{b}$, $\vec{c}$. Check up that the arguments hold even when one or both of these angles is obtuse.

Because of the commutativity of scalar multiplication, we also have

$$(\vec{a}+\vec{b}) \cdot \vec{c} = \vec{a} \cdot \vec{c} + \vec{b} \cdot \vec{c}.$$

Remarks

Addition of vectors being associative, the sum of n vectors $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$ is well defined. By finite induction on n the result about distributivity of the scalar multiplication over addition can be generalized to

$$\vec{a} \cdot (\vec{a}_1 + \vec{a}_2 + \ldots + \vec{a}_n) = \vec{a} \cdot \vec{a}_1 + \vec{a} \cdot \vec{a}_2 + \ldots + \vec{a} \cdot \vec{a}_n.$$

Also

$$(\vec{a}+\vec{b}) \cdot (\vec{c}+\vec{d}) = \vec{a} \cdot (\vec{c}+\vec{d}) + \vec{b} \cdot (\vec{c}+\vec{d})$$

$$= \vec{a} \cdot \vec{c} + \vec{a} \cdot \vec{d} + \vec{b} \cdot \vec{c} + \vec{b} \cdot \vec{d}$$

noting that there is no need to put brackets since addition of vectors is associative. In particular,

$$\begin{aligned}(\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b}) &= (\vec{a}+\vec{b})^2\\ &= \vec{a}\cdot\vec{a}+\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{a}+\vec{b}\cdot\vec{b}\\ &= \vec{a}^{\,2}+2\vec{a}\cdot\vec{b}+\vec{b}^{\,2}\end{aligned}$$

since scalar multiplication of vectors is commutative.

Note, however, that $(\vec{a}\cdot\vec{b})\cdot\vec{c}$, $\vec{a}\cdot(\vec{b}\cdot\vec{c})$ are meaningless since the scalar product is defined only between two vectors. $\vec{a}\cdot\vec{b}, \vec{b}\cdot\vec{c}$ being scalars their scalar product with a vector is not defined.

Now, consider

$$\vec{a}\cdot(\vec{b}-\vec{c})$$

We know

$$\begin{aligned}\vec{a}\cdot(\vec{b}-\vec{c}) &= \vec{a}\cdot\{\vec{b}+(-\vec{c})\}\\ &= \vec{a}\cdot\vec{b}+\vec{a}\cdot(-\vec{c})\\ &= \vec{a}\cdot\vec{b}+\vec{a}\cdot(-1\vec{c})\\ &= \vec{a}\cdot\vec{b}-1\vec{a}\cdot\vec{c}\\ &= \vec{a}\cdot\vec{b}-\vec{a}\cdot\vec{c}.\end{aligned}$$

To sum up, scalar multiplication may be treated in the context of addition and subtraction of vectors like ordinary multiplication of real numbers. One should, of course, not forget to put the dots at the right places !

If now, $\alpha\vec{i}+\beta\vec{j}, \gamma\vec{i}+\delta\vec{j}$ are two vectors in a plane referred to a pair of rectangular axes, $\alpha, \beta, \gamma, \delta$ being real numbers,

$$(\alpha\vec{i}+\beta\vec{j})\cdot(\gamma\vec{i}+\delta\vec{j}) = \alpha\gamma\,\vec{i}\cdot\vec{i}+\alpha\delta\,\vec{i}\cdot\vec{j}+\beta\gamma\,\vec{j}\cdot\vec{i}+\beta\delta\,\vec{j}\cdot\vec{j}$$

using distributivity of the scalar product with reference to addition. We have seen earlier that

$$\vec{i}\cdot\vec{i}=\vec{j}\cdot\vec{j}=1 \quad\text{while}\quad \vec{i}\cdot\vec{j}=\vec{j}\cdot\vec{i}=0.$$

So

$$(\alpha\vec{i}+\beta\vec{j})\cdot(\gamma\vec{i}+\delta\vec{j}) = \alpha\gamma+\beta\delta.$$

Thus, the rule of obtaining the scalar product in this case is to multiply the corresponding scalar coefficients and add. This applies to vectors in 3-dimensional space too. For example

$$(\alpha_1\vec{i}+\alpha_2\vec{j}+\alpha_3\vec{k})\cdot(\alpha'_1\vec{i}+\alpha'_2\vec{j}+\alpha'_3\vec{k}) = \alpha_1\alpha'_1+\alpha_2\alpha'_2+\alpha_3\alpha'_3.$$

We found earlier that the angle θ between the two vectors $\vec{a}, \vec{b} \neq \vec{0}$ is given by

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\,|\vec{b}|}.$$

If $\vec{a} = \alpha\,\vec{i} + \beta\vec{j}$, $\vec{b} = \vec{\gamma} i + \delta\,\vec{j}$, then

$$\cos\theta = \frac{\alpha\gamma + \beta\delta}{\sqrt{\alpha^2+\beta^2}\sqrt{\gamma^2+\delta^2}}$$

If $\alpha_1\vec{i} + \alpha_2\vec{j} + \alpha_3\vec{k}$, $\alpha'_1\vec{i} + \alpha'_2\vec{j} + \alpha'_3\vec{k}$ are vectors in 3-dimensional space, we have then for the angle θ between them

$$\cos\theta = \frac{\alpha_1\alpha'_1 + \alpha_2\alpha'_2 + \alpha_3\alpha'_3}{\sqrt{\alpha_1^2+\alpha_2^2+\alpha_3^2}\,\sqrt{\alpha_1'^2+\alpha_2'^2+\alpha_3'^2}}.$$

Example 9.15

If a, b, c are the lengths of the sides opposite respectively to the angles A, B, C of a triangle ABC, show that

$$\cos C = \frac{a^2+b^2-c^2}{2ab}$$

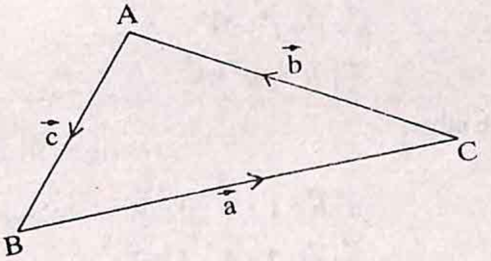


Fig. 9.43

Solution

Let $\vec{a}, \vec{b}, \vec{c}$ be the vectors $\overrightarrow{BC}$, $\overrightarrow{CA}$, $\overrightarrow{AB}$ respectively as in Fig. 9.43.

Then
$$\overrightarrow{BA} = -\overrightarrow{AB} = -\vec{c}$$
$$= \vec{a} + \vec{b}.$$

So
$$(-\vec{c})\cdot(-\vec{c}) = (\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b})$$
$$= \vec{a}^2 + 2\vec{a}\cdot\vec{b} + \vec{b}^2$$

$$= a^2 + 2ab \cos(\pi - C) + b^2$$

i.e.,
$$c^2 = a^2 - 2ab \cos C + b^2$$

since the square of a vector is the square of its magnitude as observed earlier. Hence

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

Remark

The above formula is known as a cosine formula for a triangle. We have the companion cosine formulae:

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos B = \frac{c^2 + a^2 - b^2}{2ca}$$

Example 9.16

Prove that the vectors

$$\vec{a} = \vec{i} + 2\vec{j} + \vec{k}$$
$$\vec{b} = \vec{i} + \vec{j} - 3\vec{k}$$
$$\vec{c} = 7\vec{i} - 4\vec{j} + \vec{k}$$

are at right angles to each other.

Solution

$$\vec{a} \cdot \vec{b} = 1 + 2 - 3 = 0$$
$$\vec{b} \cdot \vec{c} = 7 - 4 - 3 = 0$$
$$\vec{c} \cdot \vec{a} = 7 - 8 + 1 = 0$$

so that $\vec{a}, \vec{b}, \vec{c}$ are at right angles to each other.

Example 9.17

Find the area of the triangle formed by the points

$$A = (1, 1, 1), \quad B = (1, 2, 3), \quad C = (2, 3, 1)$$

with reference to a rectangular system of axes.

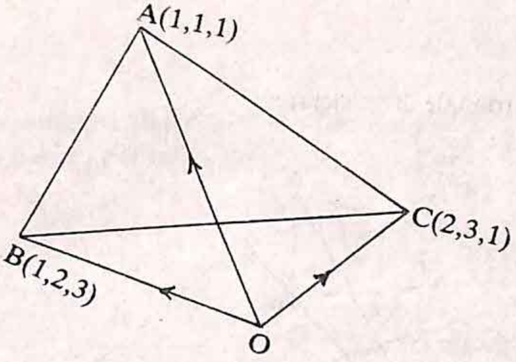


Fig. 9.44

Solution

If O is the origin (See Fig. 9.44),

$$\overrightarrow{OA} = \vec{i} + \vec{j} + \vec{k}$$
$$\overrightarrow{OB} = \vec{i} + 2\vec{j} + 3\vec{k}$$
$$\overrightarrow{OC} = 2\vec{i} + 3\vec{j} + \vec{k}.$$

So

$$\overrightarrow{BC} = \vec{i} + \vec{j} - 2\,\vec{k}$$
$$\overrightarrow{AB} = 0 \cdot \vec{i} + \vec{j} + 2\,\vec{k}$$

Hence,

$$a = BC = |\overrightarrow{BC}| = \sqrt{6}\,, \quad c = AB = |\overrightarrow{AB}| = \sqrt{5}\,.$$

The angle between $\overrightarrow{AB}$ and $\overrightarrow{BC}$ is given by

$$\cos B = \frac{\overrightarrow{AB} \cdot \overrightarrow{BC}}{|\overrightarrow{AB}|\ |\overrightarrow{BC}|} = \frac{-3}{\sqrt{6}\sqrt{5}} = \frac{-3}{\sqrt{30}}\,.$$

So

$$\sin B - \sqrt{1 - \cos^2 B} = \frac{\sqrt{7}}{\sqrt{10}}.$$

The area of $\Delta\, ABC$

$$= \frac{1}{2}\, ac \sin B = \frac{\sqrt{5}\,\sqrt{6}}{2} \cdot \frac{\sqrt{7}}{\sqrt{10}}$$
$$= \frac{1}{2}\sqrt{21}\,.$$

Example 9.18

Show that the altitudes of a triangle are concurrent.

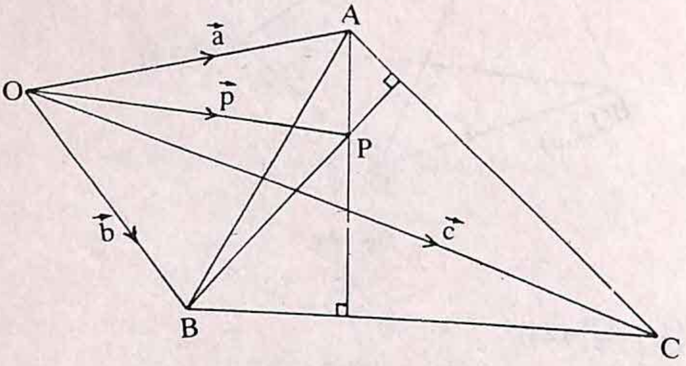


Fig. 9.45

Solution

Choose any point O of the plane of the triangle as the origin of vectors as in Fig. 9.45. Let ABC be the triangle, the position vectors of whose vertices A, B, C are $\vec{a}, \vec{b}, \vec{c}$ respectively. Let P be the point of intersection of the altitudes through A and B and let $\vec{p}$ be the position vector of P.

Now, $\overrightarrow{BC} = \vec{c} - \vec{b}, \overrightarrow{AP} = \vec{p} - \vec{a}$ and $\overrightarrow{AP} \perp \overrightarrow{BC}$

so that

$$(\vec{p} - \vec{a}) \cdot (\vec{c} - \vec{b}) = 0$$

Similarly, $\overrightarrow{BP} \perp \overrightarrow{CA}$ so that

$$(\vec{p} - \vec{b}) \cdot (\vec{a} - \vec{c}) = 0.$$

Adding both these equations, we obtain

$$\vec{p} \cdot (\vec{c} - \vec{b} + \vec{a} - \vec{c}) - \vec{a} \cdot (\vec{c} - \vec{b}) - \vec{b} \cdot (\vec{a} - \vec{c}) = 0$$

i.e.,
$$\vec{p} \cdot (\vec{a} - \vec{b}) - \vec{a} \cdot \vec{c} + \vec{b} \cdot \vec{c} = 0$$

i.e.,
$$\vec{p} \cdot (\vec{a} - \vec{b}) - \vec{c} \cdot (\vec{a} - \vec{b}) = 0$$

i.e.,
$$(\vec{p} - \vec{c}) \cdot (\vec{a} - \vec{b}) = 0$$

i.e.,
$$\overrightarrow{CP} \cdot \overrightarrow{BA} = 0.$$

This shows that

$$\overrightarrow{PC} \perp \overrightarrow{AB}$$

Example 9.19

A tetrahedron is a three dimensional object with four faces in the form of triangles as shown in Fig. 9.46. Prove that if in a tetrahedron any two pairs of opposite edges are perpendicular, then so is the other pair.

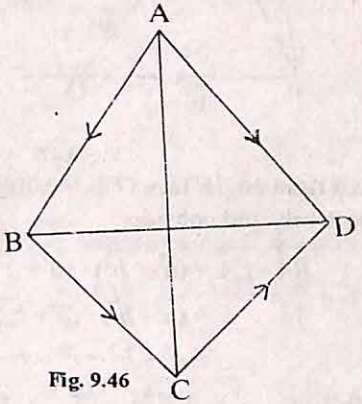


Fig. 9.46

Solution

AB, CD; AC, BD; AD, BC are pairs of opposite edges. Assume that $\overrightarrow{AB} \perp \overrightarrow{CD}$, $\overrightarrow{AD} \perp \overrightarrow{BC}$. If the position vectors of A, B, C, D with respect to some origin O are respectively $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ we have

$$(\vec{b} - \vec{a}) \cdot (\vec{d} - \vec{c}) = 0$$
$$(\vec{d} - \vec{a}) \cdot (\vec{c} - \vec{b}) = 0.$$

Adding both the equations and using associativity and distributivity we have

$$\vec{b} \cdot \vec{d} - \vec{b} \cdot \vec{c} - \vec{a} \cdot \vec{d} + \vec{a} \cdot \vec{c} + \vec{d} \cdot \vec{c} - \vec{d} \cdot \vec{b} - \vec{a} \cdot \vec{c} + \vec{a} \cdot \vec{b} = 0$$

i.e., $$\vec{d} \cdot (\vec{c} - \vec{a}) - \vec{b} \cdot (\vec{c} - \vec{a}) = 0$$

i.e., $$(\vec{c} - \vec{a}) \cdot (\vec{d} - \vec{b}) = 0$$

i.e., $$\overrightarrow{AC} \cdot \overrightarrow{BD} = 0.$$

This implies $$\overrightarrow{AC} \perp \overrightarrow{BD}$$

as required to be proved.

Example 9.20

Prove that an angle in a semicircle is a right angle.

Solution

Let BAC be a semicircle whose binding diameter is BC, O being the centre of the circle. (See Fig. 9.47).

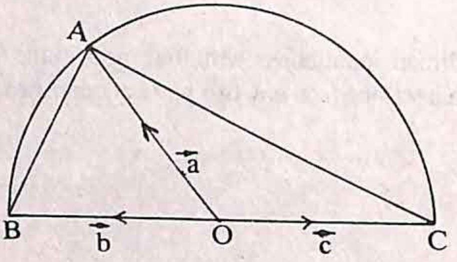


Fig. 9.47

To prove that $\angle BAC$ is a right angle, take O as the origin of vectors. Let $\vec{a}, \vec{b}, \vec{c}$ be the position vectors of A, B, C respectively and consider

$$\begin{aligned}\overrightarrow{BA}\cdot\overrightarrow{CA} &= (\vec{a}-\vec{b})\cdot(\vec{a}-\vec{c})\\ &= (\vec{a}-\vec{b})\cdot(\vec{a}+\vec{b}) \text{ (since } \overrightarrow{OB} = -\overrightarrow{OC})\\ &= a^2-b^2 = r^2-r^2 = 0\end{aligned}$$

where r is the radius of the circle. So $\overrightarrow{BA} \perp \overrightarrow{CA}$ and the result is proved.

Example 9.21

Show that the points whose position vectors are

$$\vec{a} = 4\vec{i}-3\vec{j}+\vec{k},\ \vec{b} = 2\vec{i}-4\vec{j}+5\vec{k},\ \vec{c} = \vec{i}-\vec{j}$$

form a right triangle.

Solution

Let the points representing the given position vectors be respectively A, B, C as shown in Fig. 9.48. Let O be the origin of vectors.

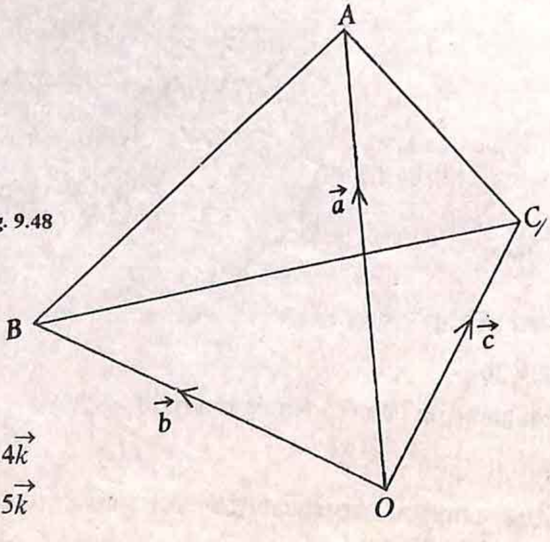


Fig. 9.48

Now, $\overrightarrow{AB} = \vec{b}-\vec{a} = -2\vec{i}-\vec{j}+4\vec{k}$

$\overrightarrow{BC} = \vec{c}-\vec{b} = -\vec{i}+3\vec{j}-5\vec{k}$

$$\overrightarrow{CA} = \vec{a} - \vec{c} = 3\vec{i} - 2\vec{j} + \vec{k}$$

Note that $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \vec{0}$. Thus, A, B, C form a triangle.

Now, $\overrightarrow{AB} \cdot \overrightarrow{CA} = (-2\vec{i} - \vec{j} + 4\vec{k}) \cdot (3\vec{i} - 2\vec{j} + \vec{k})$

$= -6 + 2 + 4 = 0$

Thus, $\angle BAC$ is right angle.

Hence, ABC is a right triangle.

EXERCISE 9.3

1. Constant forces $2\vec{i} - 5\vec{j} + 6\vec{k}$ and $-\vec{i} + 2\vec{j} - \vec{k}$ act on a particle. Determine the work done when the particle is displaced from a point A with position vector $4\vec{i} - 3\vec{j} - 2\vec{k}$ to a point B with position vector $6\vec{i} + \vec{j} - 3\vec{k}$.

2. A particle acted on by constant forces $4\vec{i} + \vec{j} - 3\vec{k}$ and $3\vec{i} + \vec{j} - \vec{k}$ is displaced from the point $\vec{i} + 2\vec{j} + 3\vec{k}$ to the point $5\vec{i} + 4\vec{j} + \vec{k}$. Find the total work done by the forces.

3. Prove that the diagonals of a rhombus are perpendicular bisectors of each other.

4. Find the work done in moving an object along the vector $3\vec{i} + 2\vec{j} - 5\vec{k}$ if the force applied is $\vec{F} = 2\vec{i} - \vec{j} - \vec{k}$.

5. If $\vec{a}$ is any vector in space, then show that

$$\vec{a} = (\vec{a} \cdot \vec{i})\vec{i} + (\vec{a} \cdot \vec{j})\vec{j} + (\vec{a} \cdot \vec{k})\vec{k}.$$

6. Show that

$$\frac{1}{7}(2\vec{i} + 3\vec{j} + 6\vec{k}), \frac{1}{7}(3\vec{i} - 6\vec{j} + 2\vec{k}), \frac{1}{7}(6\vec{i} + 2\vec{j} - 3\vec{k})$$

are mutually perpendicular unit vectors.

7. Prove that

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B.$$

[*Hint*: Take unit vectors making angles A, B with the positive side of the x-axis. Their scalar product is $\cos(A - B)$.]

8. Prove that the cosine formula for triangles is equivalent to the definition of the dot product. [*Hint*: If OAB is the triangle and $\angle OAB = \theta$ by the cosine formula,

$$AB^2 = OA^2 + OB^2 - 2\, OA \cdot OB \cos \theta.$$

But $$AB^2 = |\overrightarrow{AB}|^2 = |\overrightarrow{OB} - \overrightarrow{OA}|^2$$
$$= \overrightarrow{OB}^2 - 2\overrightarrow{OA} \cdot \overrightarrow{OB} + \overrightarrow{OA}^2 \;]$$

9. If the vertices A, B, C of ΔABC have position vectors $(1, 2, 3), (-1, 0, 0), (0, 1, 2)$ respectively, what is the magnitude of $\angle ABC$?

10. If A, B, C have position vectors $(0, 1, 1)$, $(3, 1, 5)$, $(0, 3, 3)$ respectively, show that $\Delta\, ABC$ is right angled at C.

(b) Vector Product or Cross Product of Two Vectors

Definition

If $\vec{a}, \vec{b}$ are two non-zero vectors which are not parallel, or equivalently non-collinear, their vector product $\vec{a} \times \vec{b}$, in that order, is defined to be a *vector* whose magnitude is $|\vec{a}|\,|\vec{b}|\sin\theta$ where θ is the angle between $\vec{a}$ and $\vec{b}$ as defined earlier (see p. 444) and whose direction is perpendicular to that of the plane determined by $\vec{a}$ and $\vec{b}$ in such a way that $\vec{a}, \vec{b}$ and this perpendicular direction constitute a right handed system. (See description of such a system of mutually perpendicular lines given earlier). If $\vec{n}$ is the unit vector in the direction perpendicular to $\vec{a}$ and $\vec{b}$ so that $\vec{a}, \vec{b}, \vec{n}$ form a right handed system (see Fig. 9.49),

$$\vec{a} \times \vec{b} = |\vec{a}|\,|\vec{b}|\sin\theta\,\vec{n}$$

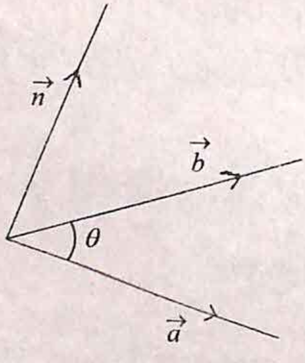


Fig. 9.49

Note that θ is *not* defined if one of $\vec{a}$ or $\vec{b}$ or both is $\vec{0}$ since $\vec{0}$ has no direction and so $\vec{n}$ is not defined. In this case we may define $\vec{a} \times \vec{b} = \vec{0}$. Again, if $\vec{a}$ and $\vec{b}$ are parallel i.e. if $\theta = 0$ or π, the direction of $\vec{n}$ is not well-defined. So in this case too the formal definition given above cannot apply. We may define $\vec{a} \times \vec{b} = \vec{0}$ in this case too. An alternative definition can be given in terms of representation of vectors as linear combinations of unit vectors in mutually perpendicular directions in a plane or three dimensional space. For the sake of illustration, let $\vec{i}, \vec{j}$ be unit vectors in two perpendicular directions in a plane.

Let
$$\vec{a} = a_1\vec{i} + a_2\vec{j}$$
$$\vec{b} = b_1\vec{i} + ib_2\vec{j},\ \ a_1, a_2, b_1, b_2 \in \mathbf{R}.$$

We define
$$\vec{a} \times \vec{b} = (a_1b_2 - a_2b_1)\vec{k}$$

where $\vec{k}$ is the unit vector perpendicular to $\vec{i}, \vec{j}$ so that $\vec{i}, \vec{j}, \vec{k}$ form a right handed system.

The angle θ between the vectors $\vec{a}$ and $\vec{b}$ both non-zero, is given by $\theta = \theta_2 - \theta_1$, where θ_1 and θ_2 are the angles made by $\vec{a}, \vec{b}$, respectively with the positive direction of the x-axis measured anticlockwise as given in Fig. 9.50. Then

$$\sin\theta = \sin(\theta_2 - \theta_1) = \sin\theta_2\cos\theta_1 - \cos\theta_2\sin\theta_1$$
$$= \frac{b_2}{|\vec{b}|}\cdot\frac{a_1}{|\vec{a}|} - \frac{a_2}{|\vec{a}|}\cdot\frac{b_1}{|\vec{b}|}$$
$$= \frac{a_1b_2 - a_2b_1}{|\vec{a}|\,|\vec{b}|}.$$

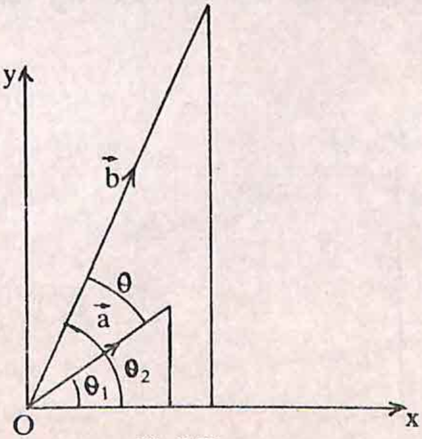


Fig. 9.50

Thus, the definition given as alternative reduces to

$$\vec{a} \times \vec{b} = |\vec{a}\,|\,|\vec{b}\,| \sin \theta\, \vec{k},$$

the positive or negative sign for $\sin \theta$ determining whether $\vec{a}, \vec{b}, \vec{k}$ form a right handed system as such or not (or rather deciding the direction of $\vec{a} \times \vec{b}$ normal to the plane of $\vec{a}$ and $\vec{b}$). This is obviously the definition for $\vec{a} \times \vec{b}$ with which we started for $\vec{a}, \vec{b} \neq \vec{0}$. But the advantage in the present definition is that it covers the case $\vec{a}$ or $\vec{b}$ or both being $\vec{0}$, which corresponds to $a_1, a_2 = 0$ or $b_1, b_2 = 0$ or both pairs being 0.

Recall that the zero vector $\vec{0}$ has magnitude 0 and has no direction and any vector having zero magnitude is the zero vector $\vec{0}$. Thus, if

$$\vec{a} = a_1\vec{i} + a_2\vec{j}$$

$$\vec{b} = b_1\vec{i} + b_2\vec{j}$$

and

$$\vec{a} \times \vec{b} = \vec{0}$$

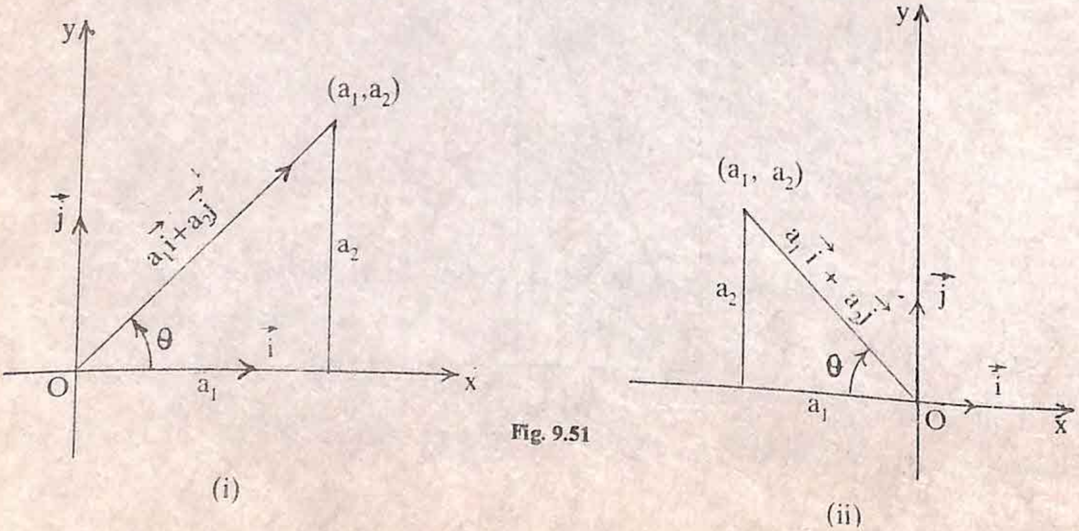
then

$$a_1b_2 - a_2b_1 = 0$$

or, equivalently,

$$\frac{b_2}{b_1} = \frac{a_2}{a_1}$$

under the assumption that $a_1, b_1 \neq 0$. We note that $\frac{a_2}{a_1}$ and $\frac{b_2}{b_1}$ are precisely $\tan \theta_1$ and $\tan \theta_2$ respectively, where θ_1 and θ_2 are the angles made by the vectors a, b respectively with the positive direction of the x-axis. Note, in this connection, that even the sign of the trigonometric ratio $\tan \theta$ is taken care of by the sign of a_1, a_2 in $a_1\vec{i} + a_2\vec{j}$ (See Fig. 9.51).



(i)

(ii)

Fig. 9.51

In the case left out, we have either (i) $a_1 = b_1 = 0$ or (ii) one of $a_1, b_1 = 0$.

In case (i) $\vec{a} = a_2\vec{j}, \vec{b} = b_2\vec{j}$ so that $\vec{a}, \vec{b}$ are along the y-axis in the same or opposite directions according as a_2, b_2 have the same or opposite signs.

In case (ii), let for definiteness $a_1 = 0$. Then $\vec{a} = a_2\vec{j}$. Let $\vec{b} = b_1\vec{i} + b_2\vec{j}$. By definition

$$\vec{a} \times \vec{b} = -a_2 b_1 \vec{k}$$

and if $\vec{a} \times \vec{b} = \vec{0}$ either $a_2 = 0$ or $b_1 = 0$. If $a_2 = 0$, then $\vec{a} = \vec{0}$. If, on the other hand, $b_1 = 0, \vec{b} = b_2\vec{j}$. Since $\vec{a} = a_1\vec{j}$, it follows that $\vec{a}, \vec{b}$ are parallel. Thus, we have the result:

Result

If $\vec{a}, \vec{b}$ are *non-zero* vectors and $\vec{a} \times \vec{b} = \vec{0}$, then $\vec{a}$ and $\vec{b}$ are parallel.

If $\vec{a}, \vec{b}$ are vectors such that $\vec{a} \times \vec{b} = \vec{0}$, then one of the following occurs:

(i) $a, b \neq 0$ and $\vec{a}, \vec{b}$ are parallel

(ii) $a = 0, b \neq 0; a \neq 0, b = 0$

(iii) $a, b = 0$

These observations lead us to the conclusion that:

If $\vec{a}$ and $\vec{b}$ are two non-zero vectors such that $\vec{a} \times \vec{b} = \vec{0}$, then $\vec{a}$ and $\vec{b}$ are either in the same direction or exactly opposite to each other.

The second alternative in the above assertion is illustrated in Fig. 9.52. To sum up, in the same way as the vanishing of $\vec{a} \cdot \vec{b}$ helps to conclude the orthogonality of $\vec{a}$ and $\vec{b}_1$ the vanishing of $\vec{a} \times \vec{b}$ helps to conclude that $\vec{a}$ and $\vec{b}$ are parallel. This result is true for 3-dimensional vectors too.

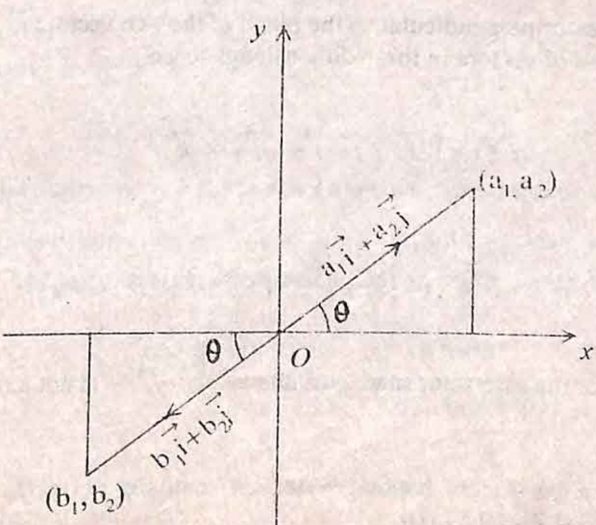


Fig. 9.52

We now establish some important properties of the vector product:

(i) *Vector multiplication is NOT commutative*

i.e., $\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}$;

however $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a} = \vec{b} \times (-\vec{a})$.

To deal with both the two dimensional and three dimensional cases together let us take the definition.

$$\vec{a} \times \vec{b} = |\vec{a}\,|\,|\vec{b}\,|\sin\theta\,\vec{k}$$

where $\vec{a}, \vec{b}, \vec{k}$ form a right handed system of vectors in three dimensions and $|\vec{k}\,| = 1, \theta$ being the angle between $\vec{a}$ and $\vec{b}$. Note that the angle between $\vec{b}$ and $\vec{a}$ is then $2\pi - \theta$ and $\sin(2\pi - \theta) = -\sin\theta$. $\vec{b}, \vec{a}, \vec{k}$ is again a right handed system. So

$$\vec{b} \times \vec{a} = |\vec{a}\,|\;|\vec{b}\,|(-\sin\theta)\,\vec{k}.$$

Hence

$$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a} \neq \vec{b} \times \vec{a}.$$

By the definition of $\vec{a} \times \vec{b}$ for two non-zero vectors $\vec{a}, \vec{b}$, it follows that if $\vec{i}, \vec{j}$ are unit vectors in mutually perpendicular directions in a plane, then

$$\vec{i} \times \vec{i} = (-\vec{i}\,) \times \vec{i} = \vec{i} \times (-\vec{i}\,) = \vec{0}$$

$$= \vec{j} \times \vec{j} = (-\vec{j}\,) \times \vec{j} = \vec{j} \times (-\vec{j}\,)$$

while if $\vec{k}$ is a unit vector perpendicular to the plane of the two vectors $\vec{i}, \vec{j}$ such that $\vec{i}, \vec{j}, \vec{k}$ is a right handed system of vectors in three dimensional space,

$$\vec{i} \times \vec{j} = \vec{k}$$

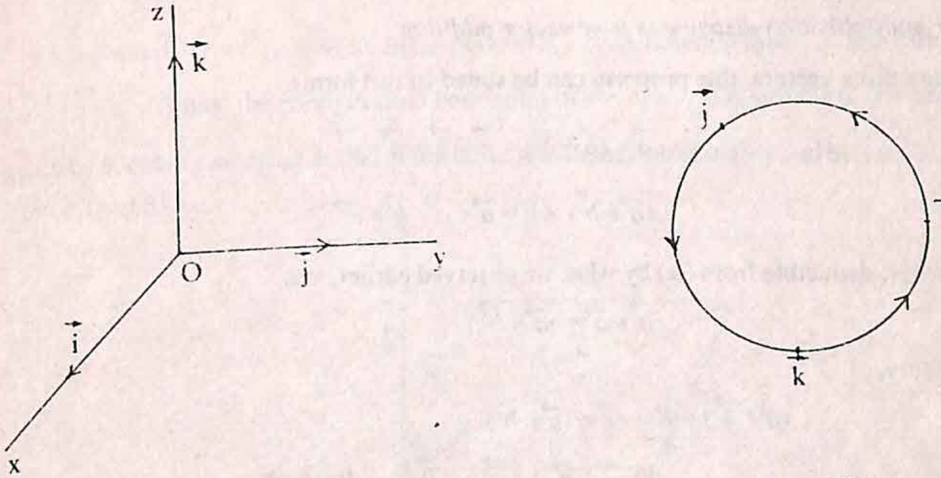
$$\vec{j} \times \vec{i} = (-\vec{i}\,) \times \vec{j} = \vec{i} \times (-\vec{j}\,) = -\vec{k}\,.$$

Also

$$(-\vec{i}\,) \times (-\vec{j}\,) = \vec{k}\,, \quad (-\vec{j}\,) \times (-\vec{i}\,) = -\vec{k}.$$

Note in the context of the assertions made just above $-\vec{i}, -\vec{j}, -\vec{k}$ is not a right handed system in space.

Now, let $\vec{i}, \vec{j}, \vec{k}$ be a right handed system of mutually perpendicular vectors in three dimensional space [See Fig. 9.53 (i)].



(i) (ii)

Fig. 9.53

Then

$$\vec{i} \times \vec{j} = \vec{k}, \quad \vec{j} \times \vec{k} = \vec{i}, \quad \vec{k} \times \vec{i} = \vec{j}$$

We note that a cyclic permutation of $\vec{i}, \vec{j}, \vec{k}$ as indicated in Fig. 9.53 (ii) does not affect the right handed nature of the system. It is an easy exercise to compute the other vector products, e.g.

$$\vec{i} \times \vec{i} = \vec{0}, \quad \vec{k} \times \vec{j} = -\vec{i}.$$

(ii) *If m is a scalar and $\vec{a}, \vec{b}$ are vectors, then*

$$(m\vec{a}) \times \vec{b} = \vec{a} \times (m\vec{b}) = m(\vec{a} \times \vec{b}).$$

What is needed to obtain this property is the fact that $\vec{a}$ and $m\vec{a}$ have the same or opposite directions according as m is positive or negative. $m = 0$ makes $m\vec{a} = \vec{0}$ and the property is then trivial. With these observations the property is easily established. For instance, when $m > 0$

$$(m\vec{a}) \times \vec{b} = |m\vec{a}|\,|\vec{b}| \sin\theta\,\vec{k}$$
$$= m\,|\vec{a}|\,|\vec{b}| \sin\theta\,\vec{k} = m(\vec{a} \times \vec{b})$$

where θ is the angle between $\vec{a}$ and $\vec{b}$ and the unit vector $\vec{k}$ is such that $\vec{a}, \vec{b}, \vec{k}$ is a right handed system. Write the proof when $m < 0$.

(iii) *Vector multiplication distributes over vector addition*

If $\vec{a}$, $\vec{b}$, $\vec{c}$ are three vectors, this property can be stated in two forms:

(α) $$\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$$

(β) $$(\vec{a} + \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}.$$

(β) is, however, deducible from (α) by what we observed earlier, viz.

$$\vec{b} \times \vec{a} = -\vec{a} \times \vec{b}.$$

More explicitly,

$$\begin{aligned}(\vec{a} + \vec{b}) \times \vec{c} &= -\vec{c} \times (\vec{a} + \vec{b}) \\ &= -\vec{c} \times \vec{a} + (-\vec{c}) \times \vec{b} \qquad \text{[by } (\alpha)] \\ &= \vec{a} \times \vec{c} + \vec{b} \times \vec{c}.\end{aligned}$$

Thus, it suffices to prove (α). This we do in two stages. First, assume that $\vec{a}$ is perpendicular to the plane in which $\vec{b}$, $\vec{c}$ lie (Fig. 9.54).

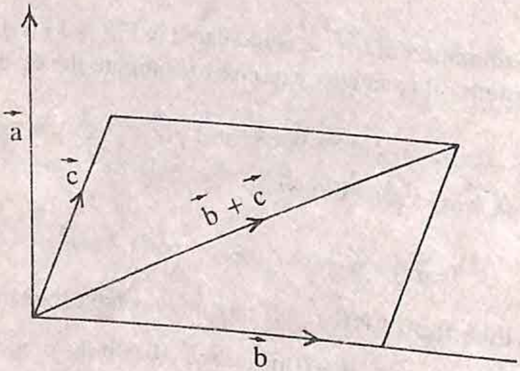


Fig. 9.54

Then, by the definition of vector product

$$\vec{a} \times \vec{b}, \quad \vec{a} \times \vec{c}, \quad \vec{a} \times (\vec{b} + \vec{c})$$

all lie in the same plane in which $\vec{b}$, $\vec{c}$, $(\vec{b} + \vec{c})$ lie. Since $\vec{a}$, $\vec{b}$ and $\vec{a}$, $\vec{c}$ are orthogonal pairs $\vec{a} \times \vec{b}$ is a vector of magnitude $|\vec{a}|\ |\vec{b}|$ in a direction in the plane of $\vec{b}$, $\vec{c}$ perpendicular to $\vec{b}$, while $\vec{a} \times \vec{c}$ is a vector of magnitude $|\vec{a}|\ |\vec{c}|$ in the same plane and in a direction perpendicular to $\vec{c}$.

Likewise $\vec{a} \times (\vec{b}+\vec{c})$ is a vector in the plane of $\vec{b}, \vec{c}$ perpendicular to $(\vec{b}+\vec{c})$ and with magnitude $|\vec{a}|\,|\vec{b}+\vec{c}|$. Thus, the configuration consisting of $\vec{a} \times \vec{b}, \vec{a} \times \vec{c}$, is similar to the parallelogram given by $\vec{b}, \vec{c}, \vec{b}+\vec{c}$ when each side of the latter parallelogram is multiplied by $|\vec{a}|$ as can be seen from Fig. 9.55.

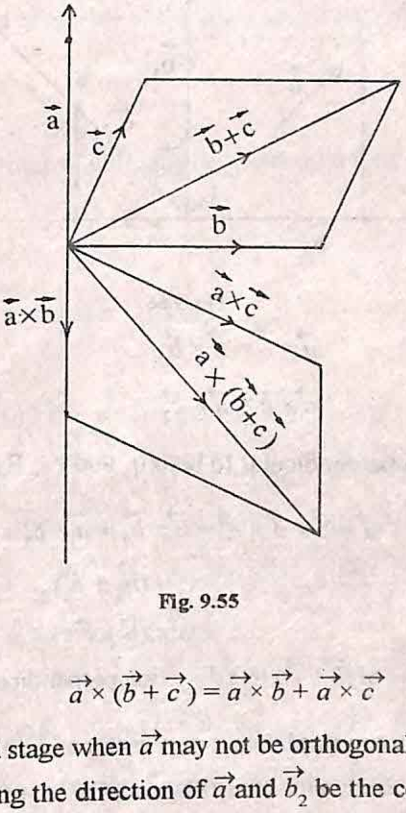


Fig. 9.55

It, therefore, follows that

$$\vec{a} \times (\vec{b}+\vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$$

We now go to the second stage when $\vec{a}$ may not be orthogonal to both $\vec{b}, \vec{c}$. In this case let $\vec{b}_1$ be the component of $\vec{b}$ along the direction of $\vec{a}$ and $\vec{b}_2$ be the component of $\vec{b}$ in a direction perpendicular to $\vec{a}$. Let $\vec{c}_1, \vec{c}_2$, be similarly the components of $\vec{c}$ in the direction of $\vec{a}$ and in a direction perpendicular to $\vec{a}$. If θ is the angle between $\vec{a}$ and $\vec{b}$, then $|\vec{b}_2| = |\vec{b}| \sin\theta$. Thus,

$$\begin{aligned} |\vec{a} \times \vec{b}| &= |\vec{a}|\,|\vec{b}| \sin\theta \\ &= |\vec{a}|\,(\,|\vec{b}| \sin\theta) \sin\frac{\pi}{2} \\ &= |\vec{a} \times \vec{b}_2| \end{aligned}$$

since the angle between $\vec{a}$ and $\vec{b}_2$ is $\frac{\pi}{2}$. Note, further, that $\vec{a}, \vec{b}_1, \vec{b}_2, \vec{b}$ lie in the same plane.

Similarly $\vec{a}, \vec{c}_1, \vec{c}_2, \vec{c}$. On this account $\vec{a} \times \vec{b}$ and $\vec{a} \times \vec{b}_2$ have the same direction as can be seen from Fig. 9.56.

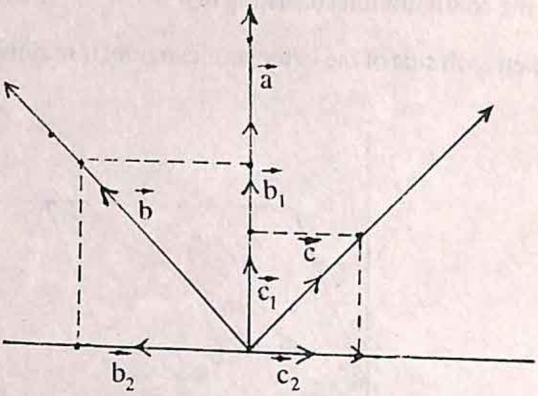


Fig. 9.56

Hence,
$$\vec{a} \times \vec{b} = \vec{a} \times \vec{b}_2.$$

Likewise
$$\vec{a} \times \vec{c} = \vec{a} \times \vec{c}_2.$$

Now, $\vec{a}$ is in a direction perpendicular to both b_2 and c_2. By the first stage of the proof

$$\begin{aligned}\vec{a} \times \vec{b} + \vec{a} \times \vec{c} &= \vec{a} \times \vec{b}_2 + \vec{a} \times \vec{c}_2 \\ &= \vec{a} \times (\vec{b}_2 + \vec{c}_2) \\ &= \vec{a} \times (\vec{b} + \vec{c})\end{aligned}$$

since $\vec{b}_2 + \vec{c}_2$ is the component of $(\vec{b} + \vec{c})$ in a direction perpendicular to $\vec{a}$ *. The distributive law is thus established.

The distributivity of vector multiplication over vector addition can now be utilized to give the cross product in terms of the basic unit vectors $\vec{i}, \vec{j}$, in a plane or the basic unit vectors $\vec{i}, \vec{j}, \vec{k}$ in three-dimensional space. More explicitly let

$$\vec{a} = a_1\vec{i} + a_2\vec{j}, \quad \vec{b} = b_1\vec{i} + b_2\vec{j}$$

be two vectors with reference to basic unit vectors $\vec{i}, \vec{j}$ in their plane. Then, by distributivity

---

* If $\vec{a}, \vec{b}$ are two vectors and $\vec{b}_1$ is the projection of $\vec{b}$ in a direction perpendicular to $\vec{a}$ in the plane of $\vec{a}$ and $\vec{b}$ i.e., $|\vec{b}_1| = |\vec{b}| \sin\theta$ (θ being the angle between a and b), $\vec{a} \times \vec{b} = \vec{a} \times \vec{b}_1$, since the direction of these vectors are one and the same.

$$\begin{aligned}\vec{a}\times\vec{b} &= (a_1\vec{i}+a_2\vec{j})\times(b_1\vec{i}+b_2\vec{j})\\ &= a_1b_1\vec{i}\times\vec{i}+a_1b_2\vec{i}\times\vec{j}\\ &\quad + a_2b_1\vec{j}\times\vec{i}+a_2b_2\vec{j}\times\vec{j}\end{aligned} \qquad \ldots (9.10)$$

If $\vec{k}$ is the unit vector perpendicular to the plane of $\vec{a}$ and $\vec{b}$ such that $\vec{i}, \vec{j}, \vec{k}$ form a right handed system, then we have the right hand side of the equation (9.10) reducing to $a_1b_2\vec{k}-a_2b_1\vec{k}$. Thus,

$$\vec{a}\times\vec{b} = (a_1b_2-a_2b_1)\vec{k} \qquad \ldots (9.11)$$

which was the definition given earlier.

If, now, a right handed system of mutually perpendicular unit vectors is taken in 3-space and denoted by $\vec{i}, \vec{j}, \vec{k}$, and if

$$\vec{a} = a_1\vec{i}+a_2\vec{j}+a_3\vec{k}$$
$$\vec{b} = b_1\vec{i}+b_2\vec{j}+b_3\vec{k}$$

then

$$\begin{aligned}\vec{a}\times\vec{b} &= (a_1\vec{i}+a_2\vec{j}+a_3\vec{k})\times(b_1\vec{i}+b_2\vec{j}+b_3\vec{k})\\ &= a_1b_1\vec{i}\times\vec{i}+a_1b_2\vec{i}\times\vec{j}+a_1b_3\vec{i}\times\vec{k}\\ &\quad + a_2b_1\vec{j}\times\vec{i}+a_2b_2\vec{j}\times\vec{j}+a_2b_3\vec{j}\times\vec{k}\\ &\quad + a_3b_1\vec{k}\times\vec{i}+a_3b_2\vec{k}\times\vec{j}+a_3b_3\vec{k}\times\vec{k}\\ &= a_1b_2\vec{k}+a_1b_3(-\vec{j})+a_2b_1(-\vec{k})\\ &\quad + a_2b_3\vec{i}+a_3b_1\vec{j}+a_3b_2(-\vec{i})\\ &= (a_2b_3-a_3b_2)\vec{i}+(a_3b_1-a_1b_3)\vec{j}+(a_1b_2-a_2b_1)\vec{k}\end{aligned} \qquad \ldots (9.12)$$

The right hand side of (9.12) can be conveniently denoted by the determinant

$$\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$$

in the sense that the expansion of the determinant (treating $\vec{i}, \vec{j}, \vec{k}$ too as numbers) gives the expression on the right hand side of (9.12).

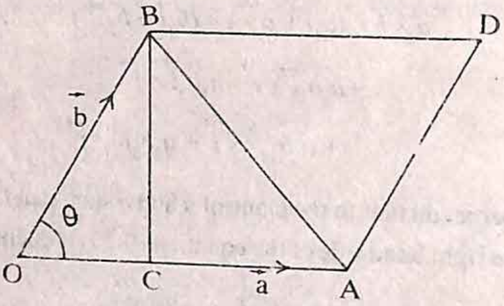


Fig. 9.57

Let $\vec{a}, \vec{b}$ be two vectors with the angle between them being θ. Let O be the origin for the vectors as shown in Fig. 9.57 and let $\overrightarrow{OA}$, $\overrightarrow{OB}$ be the vectors $\vec{a}, \vec{b}$ respectively. Then in $\Delta\, OBA$, the length of the altitude BC from B is of magnitude

$$OB \sin\theta = |b| \sin\theta.$$

Thus,

$$\begin{aligned} \text{area of triangle } OBA &= \frac{1}{2}\, OA \cdot BC \\ &= \frac{1}{2}\, |\vec{a}\,||\vec{b}\,| \sin\theta \\ &= \frac{1}{2}\, |\vec{a} \times \vec{b}\,|. \end{aligned}$$

If D is the fourth vertex of the parallelogram formed by O, B, A then its area is twice the area of the triangle OBA as is well-known.

Hence, we have the result:

Area of parallelogram with $\vec{a}, \vec{b}$ *as adjacent sides* $= |\vec{a} \times \vec{b}\,|$.

Example 9.22

Find $\vec{a} \times \vec{b}$ if $\vec{a} = 2\vec{i} + 3\vec{j} + 6\vec{k}, \vec{b} = 3\vec{i} - 6\vec{j} + 2\vec{k}.$

Solution

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & 6 \\ 3 & -6 & 2 \end{vmatrix} = (6+36)\,\vec{i} - (4-18)\,\vec{j} + (-12-9)\,\vec{k}$$

$$= 42\vec{i} + 14\vec{j} - 21\vec{k}.$$

Example 9.23

Find the angle between the vectors $(3\vec{i} + 4\vec{j})$, $(\vec{i} - \vec{j} + \vec{k})$ using their vector product.

Solution

Here, $(3\vec{i} + 4\vec{j}) \times (\vec{i} - \vec{j} + \vec{k}) = 4\vec{i} - 3\vec{j} - 7\vec{k}$

Now, $|4\vec{i} - 3\vec{j} - 7\vec{k}| = |\vec{a} \times \vec{b}| = |\vec{a}|\,|\vec{b}| \sin\theta = 5\sqrt{3}\,\sin\theta$

Therefore, $\sqrt{74} = 5\sqrt{3}\,\sin\theta$

So $\sin\theta = \dfrac{\sqrt{74}}{5\sqrt{3}}$.

Example 9.24

Find a unit vector perpendicular to both the vectors

$$\vec{\alpha} = 4\vec{i} - \vec{j} + 3\vec{k} \quad \text{and} \quad \vec{\beta} = -2\vec{i} + \vec{j} - 2\vec{k}.$$

Solution

The required unit vector is $\dfrac{\vec{\alpha} \times \vec{\beta}}{|\vec{\alpha} \times \vec{\beta}|}$. The vector

$$|\vec{\alpha} \times \vec{\beta}| = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & -1 & 3 \\ -2 & 1 & -2 \end{vmatrix} = (2-3)\vec{i} - (-8+6)\vec{j} + (4-2)\vec{k}$$

$$= -\vec{i} + 2\vec{j} + 2\vec{k}.$$

Hence, $|\vec{\alpha} \times \vec{\beta}| = \sqrt{1^2 + 2^2 + 2^2} = \sqrt{9} = 3.$

So desired unit vector is $\dfrac{1}{3}(-\vec{i} + 2\vec{j} + 2\vec{k})$.

Note : The negative of the above vector also fulfils the given requirement.

Example 9.25

If $a = \vec{i} - 2\vec{j} + \vec{k}$, $\vec{b} = 2\vec{i} - \vec{j} + \vec{k}$, $\vec{c} = \vec{i} + \vec{j} - 2\vec{k}$, compute $(\vec{a} \times \vec{b}) \times \vec{c}$ and $\vec{a} \times (\vec{b} \times \vec{c})$.

Solution

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -2 & 1 \\ 2 & -1 & 1 \end{vmatrix} = (-2+1)\vec{i} - (1-2)\vec{j} + (-1+4)\vec{k}$$

$$= -\vec{i} + \vec{j} + 3\vec{k}$$

So

$$(\vec{a}\times\vec{b})\times\vec{c}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ -1 & 1 & 3\\ 1 & 1 & -2\end{vmatrix}=(-2-3)\vec{i}-(2-3)\vec{j}+(-1-1)\vec{k}$$

$$=-5\vec{i}+\vec{j}-2\vec{k}.$$

Now,

$$\vec{b}\times\vec{c}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 2 & -1 & 1\\ 1 & 1 & -2\end{vmatrix}=(2-1)\vec{i}-(-4-1)\vec{j}+(2+1)\vec{k}$$

$$=\vec{i}+5\vec{j}+3\vec{k}$$

So

$$\vec{a}\times(\vec{b}\times\vec{c})=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 1 & -2 & 1\\ 1 & 5 & 3\end{vmatrix}=(-6-5)\vec{i}-(3-1)\vec{j}+(5+2)\vec{k}$$

$$=-11\vec{i}-2\vec{j}+7\vec{k}$$

Remark

Note that $(\vec{a}\times\vec{b})\times\vec{c}\neq\vec{a}\times(\vec{b}\times\vec{c})$. In other words, vector multiplication is *not* an associative operation.

Example 9.26

Find the area of the parallelogram whose diagonals are

$$\vec{a}=3\vec{i}+\vec{j}-2\vec{k}\text{ and }\vec{b}=\vec{i}-3\vec{j}+4\vec{k}.$$

Solution

Let *ABCD* be the parallelogram (See Fig. 9.58).

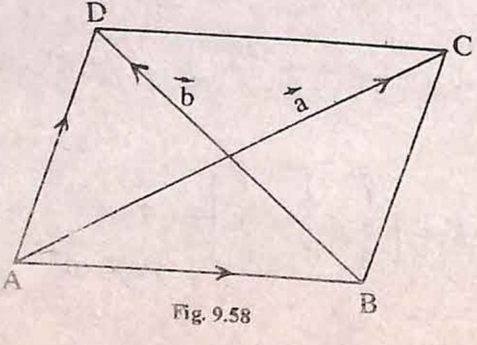


Fig. 9.58

Now

$$\overrightarrow{AB} = \frac{1}{2}\vec{a} - \frac{1}{2}\vec{b}.$$

An adjacent side

$$\overrightarrow{AD} = \frac{1}{2}\vec{a} + \frac{1}{2}\vec{b}.$$

So area of the parallelogram *ABCD*

$$= \overrightarrow{AB} \times \overrightarrow{AD}$$

$$= \left| \left(\frac{1}{2}\vec{a} - \frac{1}{2}\vec{b} \right) \times \left(\frac{1}{2}\vec{a} + \frac{1}{2}\vec{b} \right) \right|$$

$$= \frac{1}{4} \,|(\vec{a} - \vec{b}) \times (\vec{a} + \vec{b})|$$

$$= \frac{1}{4} \,|\vec{a} \times \vec{a} + \vec{a} \times \vec{b} - \vec{b} \times \vec{a} - \vec{b} \times \vec{b}\,|$$

$$= \frac{1}{4} \,|2\,\vec{a} \times \vec{b}\,| = \frac{1}{2} \,|\,\vec{a} \times \vec{b}\,|$$

$$= \frac{1}{2} \,|(4-6)\,\vec{i} - (12+2)\,\vec{j} + (-9-1)\,\vec{k}|$$

$$= \frac{1}{2} \,|-2\vec{i} - 14\vec{j} - 10\vec{k}|$$

$$= |-\vec{i} - 7\,\vec{j} - 5\,\vec{k}\,|$$

$$= (1 + 49 + 25)^{\frac{1}{2}} = (75)^{\frac{1}{2}}$$

$$= 5\sqrt{3}$$

Example 9.27

Find the area of triangle *PQR* when *P*, *Q*, *R* have respectively coordinates (1, 3, 2), (2, −1, 1), (−1, 2, 3) with reference to a rectangular system of coordinates.

Solution

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$$

$$= (2-1)\vec{i} + (-1-3)\vec{j} + (1-2)\vec{k} = \vec{i} - 4\vec{j} - \vec{k}$$

$$\overrightarrow{PR} = (-1-1)\vec{i} + (2-3)\vec{j} + (3-2)\vec{k} = -2\vec{i} - \vec{j} + \vec{k}$$

$$\text{Area of } \Delta\, PQR = \frac{1}{2}\,|\overrightarrow{PQ} \times \overrightarrow{PR}\,|$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -4 & -1 \\ -2 & -1 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\,|-5\vec{i} + \vec{j} - 9\vec{k}\,|$$

$$= \frac{1}{2}\sqrt{5^2 + 1^2 + 9^2} = \frac{1}{2}\sqrt{107}$$

Example 9.28

Prove the law of sines for a triangle.

Solution

Let $\vec{a}, \vec{b}, \vec{c}$ denote the vectors represented by the sides BC, CA, AB respectively of ΔABC (Fig. 9.59). Then we know that $\vec{a} + \vec{b} + \vec{c} = \vec{0}$. We have

$$\vec{a} \times (\vec{a} + \vec{b} + \vec{c}) = \vec{a} \times \vec{0} = \vec{0}$$

i.e.,

$$\vec{a} \times \vec{a} + \vec{a} \times \vec{b} + \vec{a} \times \vec{c} = \vec{0}$$

i.e.,

$$\vec{a} \times \vec{b} + \vec{a} \times \vec{c} = \vec{0} \qquad \text{... (9.13)}$$

Similarly,

$$\vec{b} \times \vec{a} + \vec{b} \times \vec{c} = \vec{0} \qquad \text{... (9.14)}$$

$$\vec{c} \times \vec{a} + \vec{c} \times \vec{b} = \vec{0} \qquad \text{... (9.15)}$$

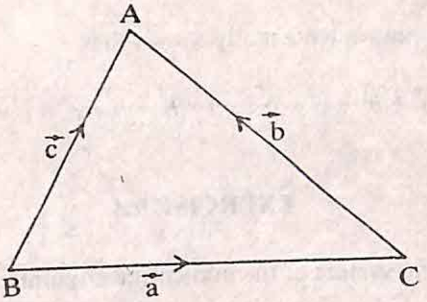


Fig. 9.59

Adding equations (9.13), (9.14) above and observing that

$$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$$

so that

$$\vec{a} \times \vec{b} + \vec{b} \times \vec{a} = \vec{0}$$

we have

$$\vec{a} \times \vec{c} + \vec{b} \times \vec{c} = \vec{0}$$

or,

$$-\vec{c} \times \vec{a} + \vec{b} \times \vec{c} = \vec{0}$$

or,

$$\vec{b} \times \vec{c} = \vec{c} \times \vec{a}.$$

We can similarly prove that

$$\vec{a} \times \vec{b} = \vec{b} \times \vec{c}.$$

So we have

$$\vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a}.$$

In particular,

$$|\vec{a} \times \vec{b}| = |\vec{b} \times \vec{c}| = |\vec{c} \times \vec{a}|.$$

So

$$|\vec{a}|\,|\vec{b}| \sin C = |\vec{b}|\,|\vec{c}| \sin A = |\vec{c}|\,|\vec{a}| \sin B.$$

With the usual notation this reduces on division by $|\vec{a}|\,|\vec{b}|\,|\vec{c}|$ and taking the reciprocal

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

which is the law of sines for a triangle.

Remark

Note that in the above example we have really shown that

$$\vec{a}+\vec{b}+\vec{c}=\vec{0} \Rightarrow \vec{a}\times\vec{b}=\vec{b}\times\vec{c}=\vec{c}\times\vec{a}.$$

EXERCISE 9.4

1. If $\vec{a}, \vec{b}, \vec{c}$ are the position vectors of the non-collinear points A, B, C respectively in space, show that

$$\vec{b}\times\vec{c}+\vec{c}\times\vec{a}+\vec{a}\times\vec{b}$$

is perpendicular to plane ABC.

2. Find the area of the parallelogram having two adjacent sides OA and OB where O is the origin and the position vectors of A, B are respectively

$$\vec{a}=\vec{i}+2\vec{j}+3\vec{k}, \quad \vec{b}=-3\vec{i}-2\vec{j}+\vec{k}.$$

3. Show that

$$(\vec{a}-\vec{b})\times(\vec{a}+\vec{b})=2(\vec{a}\times\vec{b}).$$

Interpret the result geometrically.

4. Given

$$\vec{a}=\frac{1}{7}(2\vec{i}+3\vec{j}+6\vec{k}), \vec{b}=\frac{1}{7}(3\vec{i}-6\vec{j}+2\vec{k}), \vec{c}=\frac{1}{7}(6\vec{i}+2\vec{j}-3\vec{k}),$$

$\vec{i}, \vec{j}, \vec{k}$ being a right handed orthogonal system of unit vectors in space, show that $\vec{a}, \vec{b}, \vec{c}$ is also another such system.

5. If with reference to a right handed system of mutually perpendicular unit vectors $\vec{i}, \vec{j}, \vec{k}$,

$$\vec{\alpha}=3\vec{i}-\vec{j}, \quad \vec{\beta}=2\vec{i}+\vec{j}-3\vec{k}$$

express $\vec{\beta}$ in the form $\vec{\beta}=\vec{\beta}_1+\vec{\beta}_2$ where $\vec{\beta}_1$ is parallel to $\vec{\alpha}$ and $\vec{\beta}_2$ is perpendicular to $\vec{\alpha}$.

9.4 Moment of Force about a Point

Let $\vec{F}$ be the force vector and let P be a point on the line of action of $\vec{F}$. Let $\vec{r}$ be the position vector of P with respect to the point O.

The moment of $\vec{F}$ about O is defined to be the vector

$$\vec{M}=\vec{r}\times\vec{F} \qquad \ldots(9.16)$$

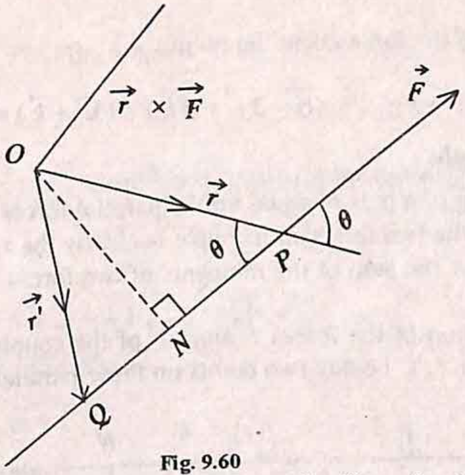


Fig. 9.60

The moment is independent of the position of P on the line of action of $\vec F$. This can be proved as follows:

Let Q be any other point on the line of action of $\vec F$ (See Fig. 9.60) and $\overrightarrow{OQ} = \vec r'$

$$\text{Then } \vec M = \vec r' \times \vec F = (\vec r + \overrightarrow{PQ}) \times \vec F$$
$$= \vec r \times \vec F + \overrightarrow{PQ} \times \vec F$$
$$= \vec r \times \vec F + \vec 0 = \vec r \times \vec F$$
$$(\overrightarrow{PQ} \times \vec F = \vec 0 \text{ as } \overrightarrow{PQ} \text{ and } \vec F \text{ are collinear vectors})$$

The magnitude of the moment $\vec M$ is

$$|\vec F|\,|\vec r|\sin\theta = |\vec F|\,|\vec r|\sin\angle OPN = |\vec F|\,ON,$$

where ON is the perpendicular from O on the line of action of the force. The direction of the vector moment $\vec M$ is along the normal to the plane OPN such that $\vec r$, $\vec F$ and $\vec M$ form a right handed system.

If the force $\vec F$ is the resultant of a number of forces $\vec F_1, \vec F_2, \ldots$ acting at the same point of application, P where $\overrightarrow{OP} = \vec r$ then it can be shown that the moment of $\vec F$ about O is given by

$$\vec M = \vec r \times \vec F = \vec r \times (\vec F_1 + \vec F_2 + \ldots) = \vec r \times \vec F_1 + \vec r \times \vec F_2 + \ldots$$

This gives us the generalized theorem of moment viz.

The algebraic sum of the moments of a system of forces acting at a point about any point is equal to the moment of their resultant about the same point.

Example 9.29

Find the moment about the point $\vec i + 2\vec j - \vec k$ of a force represented by $3\vec i + \vec k$ acting through the point $2\vec i - \vec j + 3\vec k$.

Solution

Here the force $\vec F = 3\vec i + \vec k$

$$\vec r = (2\vec i - \vec j + 3\vec k) - (\vec i + 2\vec j - \vec k) = \vec i - 3\vec j + 4\vec k$$

The required moment of the force about the point

$$(\vec{i}+2\vec{j}-\vec{k}) = \vec{r}\times\vec{F} = (\vec{i}-3\vec{j}+4\vec{k})\times(3\vec{i}+\vec{k}) = -3\,\vec{i}.+11\vec{j}+9\vec{k}$$

9.5 Moment of a Couple

A system consisting of a pair of equal unlike parallel forces is called a *couple*.

The vector sum of the two forces of a couple is clearly the zero vector.

It can be shown that the sum of the moments of two forces of a couple is the same about every point.

Let the lines of action of the forces $\vec{F}$ and $-\vec{F}$ of the couple be the two parallel lines as shown in Fig. 9.61. Let A, C be any two points on these parallel lines.

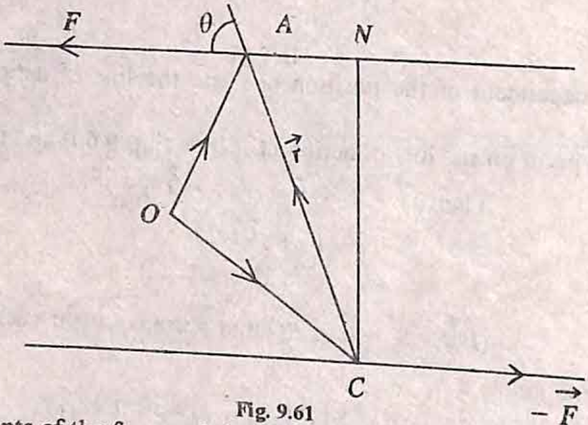


Fig. 9.61

The sum of the moments of the forces of the couple about any point O is given by

$$= \overrightarrow{OA}\times\vec{F}+\overrightarrow{OC}\times(-\vec{F})$$

$$= (\overrightarrow{OA}-\overrightarrow{OC})\times\vec{F} = \overrightarrow{CA}\times\vec{F}.$$

This shows that the sum of the moments of the forces of a given couple is independent of the point O. Again,

$$|\vec{M}| = |\overrightarrow{CA}\times\vec{F}|$$

$$= |\vec{F}|\,|\overrightarrow{CA}|\,|\sin\theta|,$$ where θ is the angle between $\overrightarrow{CA}$ and $\vec{F}$

$$= |\vec{F}|\,CN$$

This constant sum of the moments of the forces of a couple is the moment of the couple.

Any line perpendicular to the plane of a couple is called the axis of the couple. So the axis of a couple is parallel to its moment vector.

Thus, the moment of a couple is a vector perpendicular to the plane of the couple and its magnitude is the product of the magnitude of either force with the perpendicular distance between the lines of the forces.

Example 9.30

Two unlike forces of equal magnitudes $3\vec{i}+\vec{k}$ and $-3\vec{i}-\vec{k}$ are acting at the points $\vec{i}+2\vec{j}-\vec{k}$

and $2\vec{i}-\vec{j}+3\vec{k}$ respectively. Find the moment of the couple formed by these forces.

Solution

Moment of the couple $=(\vec{r}_1-\vec{r}_2)\times\vec{F}$, where $\vec{r}_1$ and $\vec{r}_2$ are position vectors of the points of application of the forces $\vec{F}$ and $-\vec{F}$.

Here,

$$\vec{r}_1-\vec{r}_2 = (\vec{i}+2\vec{j}-\vec{k})-(2\vec{i}-\vec{j}+3\vec{k})$$
$$= -\vec{i}+3\vec{j}-4\vec{k}$$

Also.

$$\vec{F} = 3\vec{i}+\vec{k}$$

Therefore, moment of the couple

$$=(-\vec{i}+3\vec{j}-4\vec{k})\times(3\vec{i}+\vec{k})$$
$$=3\vec{i}-11\vec{j}-9\vec{k}.$$

EXERCISE 9.5

1. Find the moment about $(1, -1, -1)$ of the force $3\vec{i}+4\vec{j}-5\vec{k}$ acting at $(1, 0, -2)$.
2. The force represented by $3\vec{i}+2\vec{k}$ is acting through the point $5\vec{i}+4\vec{j}-3\vec{k}$. Find its moment about the point $\vec{i}+3\vec{j}+\vec{k}$.
3. Find the moment about the point $\vec{i}+2\vec{j}-\vec{k}$ of a force represented by $\vec{i}+2\vec{j}+\vec{k}$ acting through the point $2\vec{i}+3\vec{j}+\vec{k}$.
4. The force represented by $5\vec{i}+\vec{k}$ is acting through the point $9\vec{i}-\vec{j}+2\vec{k}$. Find the moment about the point $3\vec{i}-2\vec{j}+\vec{k}$.
5. Two unlike forces of equal magnitudes $\vec{j}+2\vec{k}$ and $-\vec{j}-2\vec{k}$ are acting at the point whose position vectors are given by $\vec{i}+\vec{j}+\vec{k}$ and $\vec{i}+2\vec{j}+3\vec{k}$ respectively. Find the moment of the couple formed by these forces.
6. Find the moment of the couple formed by the forces $5\vec{i}+\vec{k}$ and $-5\vec{i}-\vec{k}$ acting at the points $(9, -1, 2)$ and $(3, -2, 1)$ respectively.

9.6 Triple Products

Let $\vec{a}, \vec{b}, \vec{c}$ be three vectors in three-dimensional space. We have seen that the scalar product can be defined only between two vectors. Thus, $\vec{a} \cdot (\vec{b} \cdot \vec{c})$, $(\vec{a} \cdot \vec{b}) \cdot \vec{c}$ are meaningless symbols while $\vec{a} \cdot (\vec{b} \times \vec{c})$, $(\vec{a} \times \vec{b}) \cdot \vec{c}$ are meaningful; of course, $(\vec{a} \times \vec{b}) \times \vec{c}$, $\vec{a} \times (\vec{b} \times \vec{c})$ are also meaningful. On the other hand $\vec{a} \times (\vec{b} \cdot \vec{c})$, $(\vec{a} \cdot \vec{b}) \times \vec{c}$ are meaningless since the vector product is defined only between two vectors. Note, in this context, that $\vec{b} \cdot \vec{c}$, $\vec{a} \cdot \vec{b}$ are scalars. This leaves us to consider products of the form

$$\vec{a} \times (\vec{b} \times \vec{c}\,), \quad \vec{a} \cdot (\vec{b} \times \vec{c}) \qquad \text{... (9.17)}$$

Products of the form

$$(\vec{a} \times \vec{b}\,) \times \vec{c}, \quad (\vec{a} \times \vec{b}\,) \cdot \vec{c}$$

get reduced respectively to

$$(-\vec{c}\,) \times (\vec{a} \times \vec{b}), \quad \vec{c} \cdot (\vec{a} \times \vec{b}\,)$$

because of the properties of the vector and scalar products. These are again of the form of (9.17). Thus, it suffices to consider the *triple products* as they are called, in (9.17).

We first consider $\vec{a} \times (\vec{b} \times \vec{c})$. We prove

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c}\,)\vec{b} - (\vec{a} \cdot \vec{b}\,)\,\vec{c} \qquad \text{... (9.18)}$$

Assume that $\vec{a}, \vec{b}, \vec{c}$ are vectors in three-dimensional space. Note that $\vec{a} \cdot \vec{c}$, $\vec{a} \cdot \vec{b}$ are scalars so that the two terms on the right are meaningful.

To prove the formula, first write $\vec{q}$ for the left hand side of (9.18) and note that $\vec{q}$ is perpendicular to both $\vec{a}$ and $\vec{b} \times \vec{c}$. Since $\vec{b} \times \vec{c}$ is perpendicular to the plane of $\vec{b}$ and $\vec{c}$, it follows that $\vec{q}$ lies in the plane of $\vec{b}$ and $\vec{c}$. Thus, $\vec{q}, \vec{b}, \vec{c}$ lie in a plane. This means that they are linearly dependent (i.e. there exists a linear combination of these vectors which equals zero). So there exists x and y such that $\vec{q} = x\vec{b} + y\vec{c}$. We now choose a rectangular system of directions $\vec{i}, \vec{j}, \vec{k}$ as follows: Let $\vec{j}$ be the unit vector along $\vec{b}$. Let $\vec{k}$ be unit vector perpendicular to $\vec{b}$ and in the plane of $\vec{b}$ and $\vec{c}$. Let $\vec{i}$ be the unit vector in space such that $\vec{i}, \vec{j}, \vec{k}$ form a right handed system of unit vectors (Fig. 9.62).

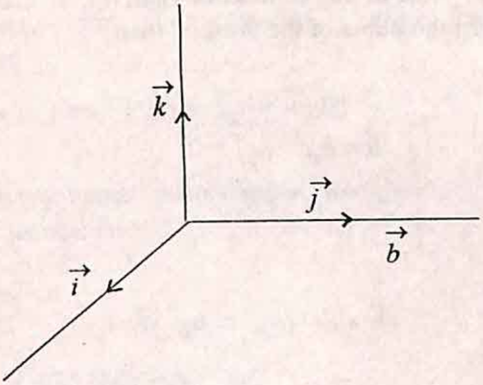


Fig. 9.62

Then

$$\begin{aligned}
\vec{b} &= b_2\vec{j}\\
\vec{c} &= c_2\vec{j} + c_3\vec{k}\\
\vec{a} &= a_1\vec{i} + a_2\vec{j} + a_3\vec{k}\\
\vec{a} \times (\vec{b} \times \vec{c}) &= (a_1\vec{i} + a_2\vec{j} + a_3\vec{k}) \times [b_2\vec{j} \times (c_2\vec{j} + c_3\vec{k})].
\end{aligned}$$

Using the properties of the vector product, clearly

$$\begin{aligned}
\vec{b} \times \vec{c} &= b_2\vec{j} \times c_3\vec{k}\\
&= b_2c_3\vec{i} \quad (\text{since } \vec{j} \times \vec{j} = \vec{0}, \vec{j} \times \vec{k} = \vec{i}).
\end{aligned}$$

So

$$\begin{aligned}
\vec{a} \times (\vec{b} \times \vec{c}) &= \vec{a} \times b_2c_3\vec{i}\\
&= (a_1\vec{i} + a_2\vec{j} + a_3\vec{k}) \times b_2c_3\vec{i}\\
&= -a_2b_2c_3\vec{k} + a_3b_2c_3\vec{j}\\
&= (a_2c_2 + a_3c_3)\, b_2\vec{j} - a_2b_2\,(c_2\vec{j} + c_3\vec{k})\\
&= (\vec{a} \cdot \vec{c})\,\vec{b} - (\vec{a} \cdot \vec{b})\,\vec{c}.
\end{aligned}$$

This proves the formula stated earlier for the case of vectors in three dimensions. The case of vectors in two dimensions can be deemed as a special case by taking these vectors as referred to two perpendicular axes in their plane together with the third axis in a direction perpendicular to their plane at the origin for the two dimensional axes so that the positive directions of the three axes form a right handed system. Note in this connection that, if $\vec{i}$, $\vec{j}$, $\vec{k}$ are the unit vectors in these directions of which $\vec{i}, \vec{j}$ are in the plane of the vectors, then

$$\begin{aligned}\vec{a} &= a_1\vec{i} + a_2\vec{j} + 0\vec{k}\\ \vec{b} &= b_1\vec{i} + b_2\vec{j} + 0\vec{k}\\ \vec{c} &= c_1\vec{i} + c_2\vec{j} + 0\vec{k}\end{aligned}$$

while

$$\vec{b} \times \vec{c} = (b_1c_2 - b_2c_1)\vec{k}.$$

Note further that

$$\begin{aligned}\vec{a} \times (\vec{b} \times \vec{c}) &= -a_1(b_1c_2 - b_2c_1)\vec{j} + a_2(b_1c_2 - b_2c_1)\vec{i}\\ &= (a_1c_1 + a_2c_2)(b_1\vec{i} + b_2\vec{j}) - (a_1b_1 + a_2b_2)(c_1\vec{i} + c_2\vec{j})\\ &= (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}.\end{aligned}$$

This proves (9.18) completely.

The triple product $\vec{a} \times (\vec{b} \times \vec{c})$ is called the *vector triple product* since it is a vector. Against this, the triple product $\vec{a} \cdot (\vec{b} \times \vec{c})$ which is a scalar, is called *scalar triple product*. Sometimes this scalar triple product is denoted by $[abc]$.

We have the following relations:

$$\left.\begin{aligned}\vec{b} \cdot (\vec{c} \times \vec{a}) &= (\vec{c} \times \vec{a}) \cdot \vec{b} = -(\vec{a} \times \vec{c}) \cdot \vec{b}\\ \vec{c} \cdot (\vec{a} \times \vec{b}) &= (\vec{a} \times \vec{b}) \cdot \vec{c} = -(\vec{b} \times \vec{a}) \cdot \vec{c}\\ \vec{a} \cdot (\vec{b} \times \vec{c}) &= (\vec{b} \times \vec{c}) \cdot \vec{a} = -(\vec{c} \times \vec{b}) \cdot \vec{a}\end{aligned}\right\} \quad \ldots (9.19)$$

which follow from the well-known properties of the scalar and vector products of two vectors.

We know $\vec{b} \times \vec{c}$ is a vector in a direction perpendicular to the plane of $\vec{b}$ and $\vec{c}$, while the dot product of two perpendicular vectors is 0. Thus, if $\vec{a}, \vec{b}, \vec{c} \neq \vec{0}$ all lie in a plane, then $\vec{a}$ and $\vec{b} \times \vec{c}$, if the latter is $\neq \vec{0}$ are perpendicular vectors so that

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = 0.$$

Even if $\vec{b} \times \vec{c} = \vec{0}$ i.e. when $\vec{b}$ and $\vec{c}$ are parallel, $\vec{a} \cdot (\vec{b} \times \vec{c}) = 0$. What was argued for $\vec{a}$ and $\vec{b} \times \vec{c}$ could have been argued for

$$\vec{b}, \vec{c} \times \vec{a}\ ;\ \vec{c}, \vec{a} \times \vec{b}.$$

Thus, $$\vec{b} \cdot (\vec{c} \times \vec{a}) = \vec{c} \cdot (\vec{a} \times \vec{b}) = 0$$

where $\vec{a}, \vec{b}, \vec{c}$ are coplanar vectors. The converse is also true provided we assume that $\vec{a}, \vec{b}, \vec{c} \neq \vec{0}$. Thus, we have

*Three vectors** $\vec{a}, \vec{b}, \vec{c}$ *are coplanar if and only if* $\vec{a} \cdot (\vec{b} \times \vec{c}) = 0$. *If this is so* $\vec{b} \cdot (\vec{c} \times \vec{a})$, $\vec{c} \cdot (\vec{a} \times \vec{b}) = 0$ *too.*

Note, therefore, that for two dimensional vectors the scalar triple product is not interesting since it is always 0. In the three dimensional case note that $\vec{a} \cdot (\vec{b} \times \vec{c}) = 0$ precisely in one of the following cases:

(i) $\vec{a}, \vec{b}, \vec{c}$ are coplanar;
(ii) one of $\vec{a}, \vec{b}, \vec{c}$ is $\vec{0}$;
(iii) $\vec{b}, \vec{c}$ are in the same or opposite directions.

Geometrical Interpretation of the Scalar Triple Product

We can get a parallelopiped having the three non-coplanar vectors $\vec{a}, \vec{b}, \vec{c}$ as edges meeting at a point as shown in Fig. 9.63.

Now, $|\vec{a} \times \vec{b}|$ = area of the parallelogram with $\vec{a}, \vec{b}$ as adjacent sides and the direction of $\vec{a} \times \vec{b}$ is normal to the plane of the vectors $\vec{a}, \vec{b}$. Thus,

$$(\vec{a} \times \vec{b}) \cdot \vec{c} = |\vec{a} \times \vec{b}|\,|\vec{c}| \cos \theta$$

where θ is the angle between $\vec{c}$ and the normal to the plane of $\vec{a}, \vec{b}$. In such a case $|\vec{c}| \cos \theta$ will be precisely the height of the parallelopiped orthogonal to its base containing $\vec{a}, \vec{b}$. Thus,

$$\begin{aligned}(\vec{a} \times \vec{b}) \cdot \vec{c} &= \text{Base area of the parallelopiped} \\ &\qquad \times \text{height of the parallelopiped on this base} \\ &= \text{Volume of the parallelopiped having the vectors} \\ &\qquad \vec{a}, \vec{b}, \vec{c} \text{ along edges meeting at a point}\end{aligned}$$

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*Note that if $\vec{a}, \vec{b}, \vec{c}$ are three vectors of which $\vec{b}$ and $\vec{c}$ are parallel, with our convention that the vectors can be deemed to have the same initial point $\vec{a}, \vec{b}, \vec{c}$ lie in a plane, viz. the plane determined by $\vec{a}$ and $\vec{b}$. Even in this case, besides $\vec{a} \cdot (\vec{b} \times \vec{c})$ being 0, $\vec{b} \cdot (\vec{c} \times \vec{a})$, $\vec{c} \cdot (\vec{a} \times \vec{b}) = 0$ since the normal to the plane of $\vec{a}$ and $\vec{c}$ or $\vec{a}$ and $\vec{b}$ is perpendicular to $\vec{b}$ or $\vec{c}$ as the case may be. If $\vec{a}, \vec{b}, \vec{c}$ are all parallel they evidently lie in a plane and the criterion holds.

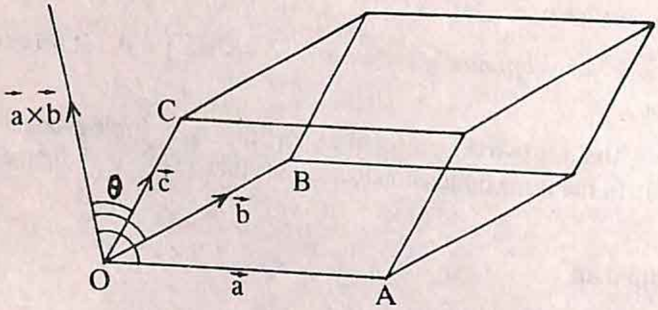

Fig. 9.63

For the above reason scalar triple product is called *box product*. It is now clear that all the products occurring in (9.19) as the first members of the chains of equalities are themselves equal. More explicitly the scalar triple product involving three vectors $\vec{a}, \vec{b}, \vec{c}$ is numerically the same in whatever way it is formed with these vectors only. In this context we can formulate the following rules for the preservation of the sign too:

1. *The sign of a dot product of the form* $\vec{a}\cdot(\vec{b}\times\vec{c})$ *is unchanged if* $\vec{a}, \vec{b}, \vec{c}$ *are cyclically permuted, i.e.*

$$\vec{a}, \vec{b}, \vec{c} \rightarrow \vec{b}, \vec{c}, \vec{a}, \quad \vec{a}, \vec{b}, \vec{c} \rightarrow \vec{c}, \vec{a}, \vec{b}.$$

2. *The sign is unchanged if the dot and cross are interchanged.* For example,

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b})\cdot\vec{c}.$$

As regards the change of sign the rule is that

3. *A breaking of a cyclical permutation of the three vectors brings in a change of sign. In a compact notation*

$$[\vec{a}\,\vec{b}\,\vec{c}\,] = [\vec{b}\,\vec{c}\,\vec{a}\,] = [\vec{c}\,\vec{a}\,\vec{b}\,]$$

while

$$[\vec{a}\;\vec{c}\;\vec{b}\,] = [\vec{c}\;\vec{b}\;\vec{a}\,] = [\vec{b}\,\vec{a}\;\vec{c}\;] = -[\vec{a}\;\vec{b}\;\vec{c}\,].$$

Note : We could omit the brackets in $\vec{a}\cdot(\vec{b}\times\vec{c})$ and just write $\vec{a}\cdot\vec{b}\times\vec{c}$ for the reason that $(\vec{a}\cdot\vec{b})\times\vec{c}$ is meaningless.

Matrix Expression for the Scalar Triple Product

Let

$$\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}, \vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}, \vec{c} = c_1\vec{i} + c_2\vec{j} + c_3\vec{k}$$

where $\vec{i}, \vec{j}, \vec{k}$ is a set of basic unit vectors in mutually perpendicular directions forming a right handed system. Then,

$$\vec{a}\cdot\vec{b}\times\vec{c} = \vec{a}\cdot\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= (a_1\vec{i} + a_2\vec{j} + a_3\vec{k})\cdot\left\{\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix}\vec{i} - \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix}\vec{j} + \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}\vec{k}\right\}$$

$$= a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

(since $\vec{i}\cdot\vec{i} = \vec{j}\cdot\vec{j} = \vec{k}\cdot\vec{k} = 1$ while $\vec{i}\cdot\vec{j} = \vec{j}\cdot\vec{k} = \vec{k}\cdot\vec{i} = 0$)

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

We have, thus, the result

$$\vec{a}\cdot\vec{b}\times\vec{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Note that $\vec{a}\cdot\vec{b}\times\vec{c} = \vec{b}\cdot\vec{c}\times\vec{a}$ obtained earlier shows that the value of the determinant is unchanged if two interchanges of rows are made therein. This is a well-known property of determinants.

Example 9.28

Show that

$$\vec{a}\times(\vec{b}\times\vec{c}) + \vec{b}\times(\vec{c}\times\vec{a}) + \vec{c}\times(\vec{a}\times\vec{b}) = \vec{0}.$$

Solution

$$\begin{aligned} \text{Left hand side} = \; & (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c} \\ & + (\vec{b}\cdot\vec{a})\vec{c} - (\vec{b}\cdot\vec{c})\vec{a} \\ & + (\vec{c}\cdot\vec{b})\vec{a} - (\vec{c}\cdot\vec{a})\vec{b} \\ = \; & \vec{0}, \end{aligned}$$

since scalar multiplication is commutative.

Note : From Example 9.28 it follows that

$$\vec{a}\times(\vec{b}\times\vec{c}),\ \ \vec{b}\times(\vec{c}\times\vec{a}),\ \ \vec{c}\times(\vec{a}\times\vec{b})$$

are linearly dependent and, therefore, lie in a plane.

Example 9.29

If

$$\vec{a}=\vec{i}-2\vec{j}-3\vec{k},\ \vec{b}=2\vec{i}+\vec{j}-\vec{k} \text{ and } \vec{c}=\vec{i}+3\vec{j}-2\vec{k},$$

find $\vec{a}\times(\vec{b}\times\vec{c})$.

Solution

$$\begin{aligned}\vec{b}\times\vec{c}&=(-2+3)\vec{i}-(-4+1)\vec{j}+(6-1)\vec{k}\\&=\vec{i}+3\vec{j}+5\vec{k}.\end{aligned}$$

So

$$\begin{aligned}\vec{a}\times(\vec{b}\times\vec{c})&=(-10+9)\vec{i}-(5+3)\vec{j}+(3+2)\vec{k}\\&=-\vec{i}-8\vec{j}+5\vec{k}.\end{aligned}$$

Example 9.30

Show that $\vec{a}\cdot(\vec{a}\times\vec{b})=0$ for any pair $\vec{a}, \vec{b}$ of vectors.

Solution

$\vec{a}\cdot\vec{a}\times\vec{b}=\vec{a}\times\vec{a}\cdot\vec{b}$ by interchanging the dot and cross. But $\vec{a}\times\vec{a}=\vec{0}$. So $\vec{a}\cdot\vec{a}\times\vec{b}=0$.

Example 9.31

Simplify

$$(\vec{b}+\vec{c})\cdot\{(\vec{c}+\vec{a})\times(\vec{a}+\vec{b})\}.$$

Solution

$$\begin{aligned}&(\vec{b}+\vec{c})\cdot\{(\vec{c}+\vec{a})\times(\vec{a}+\vec{b})\}\\&=(\vec{b}+\vec{c})\cdot(\vec{c}\times\vec{a}+\vec{c}\times\vec{b}+\vec{a}\times\vec{b}) \qquad (\text{since } \vec{a}\times\vec{a}=\vec{0})\\&=\vec{b}\cdot\vec{c}\times\vec{a}+\vec{b}\cdot\vec{c}\times\vec{b}+\vec{b}\cdot\vec{a}\times\vec{b}\\&\quad+\vec{c}\cdot\vec{c}\times\vec{a}+\vec{c}\cdot\vec{c}\times\vec{b}+\vec{c}\cdot\vec{a}\times\vec{b}\\&=\vec{b}\cdot\vec{c}\times\vec{a}+\vec{c}\cdot\vec{a}\times\vec{b}=2\,[abc]\end{aligned}$$

noting that $\vec{b} \cdot \vec{c} \times \vec{b}$ and other similar triple products are 0.

Example 9.32

If

$$\vec{a} = -3\vec{i} + 7\vec{j} + 5\vec{k}, \vec{b} = -5\vec{i} + 7\vec{j} - 3\vec{k}, \vec{c} = 7\vec{i} - 5\vec{j} - 3\vec{k}$$

are the three sides of a parallelopiped, having $\vec{a}, \vec{b}, \vec{c}$ as edges meeting at a vertex, find the volume of the parallelopiped.

Solution

The desired volume is

$$\begin{aligned} |\vec{a} \cdot \vec{b} \times \vec{c}| &= |(-3\vec{i} + 7\vec{j} + 5\vec{k}) \cdot (-36\vec{i} - 36\vec{j} - 24\vec{k})| \\ &= |108 - 252 - 120| = |-264| = 264. \end{aligned}$$

Example 9.33

If

$$\vec{a} = \vec{i} + 2\vec{j} + \vec{k}, \vec{b} = 3\vec{i} + 2\vec{j} - 7\vec{k}, \vec{c} = 5\vec{i} + 6\vec{j} - 5\vec{k},$$

show that $\vec{a}, \vec{b}, \vec{c}$ are coplanar.

Solution

$$\begin{aligned} \vec{a} \cdot \vec{b} \times \vec{c} &= (\vec{i} + 2\vec{j} + \vec{k}) \cdot (3\vec{i} + 2\vec{j} - 7\vec{k}) \times (5\vec{i} + 6\vec{j} - 5\vec{k}) \\ &= (\vec{i} + 2\vec{j} + \vec{k}) \cdot (32\vec{i} - 20\vec{j} + 8\vec{k}) \\ &= 32 - 40 + 8 = 0. \end{aligned}$$

So $\vec{a}, \vec{b}, \vec{c}$ are coplanar since $\vec{a} \neq \vec{0}$ and $\vec{b}$ and $\vec{c}$ are not parallel.

Note: The following three examples are meant as application of vectors to geometry.

Example 9.34

If a plane has the direction of its normal from origin as $\vec{n}$ and contains a point whose position vector is $\vec{a}$, show that any point in the plane has position vector $\vec{r}$ given by

$$(\vec{r} - \vec{a}) \cdot \vec{n} = 0$$

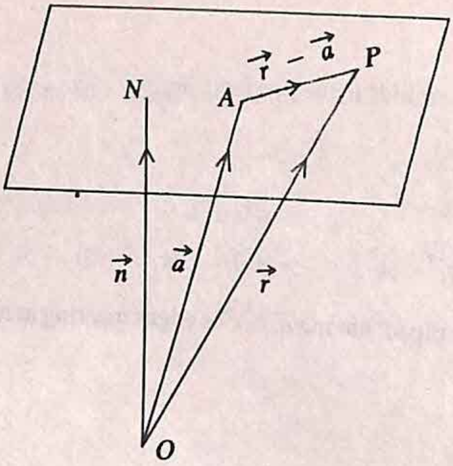

Fig. 9.64

Solution

As can be seen from Fig. 9.64, $(\vec{r}-\vec{a})$ lies in the plane and is perpendicular to $\vec{n}$ which is normal to the plane. This is expressed by the fact

$$(\vec{r}-\vec{a})\cdot\vec{n}=0 \qquad \text{... (9.20)}$$

Note : The equation (9.20) is said to be the vector equation of the plane.

Example 9.35

Find the equation satisfied by the coordinates (x, y, z) of a point on the plane which is normal to the vector $\vec{i}+\vec{j}+\vec{k}$ and passes through (2, 3, 4).

Solution

In this case

$$\vec{n}=\vec{i}+\vec{j}+\vec{k}, \vec{a}=2\vec{i}+3\vec{j}+4\vec{k} \text{ and } \vec{r}=x\vec{i}+y\vec{j}+z\vec{k}.$$

So

$$(\vec{r}-\vec{a})\cdot\vec{n}=0$$

$$\Rightarrow \quad \{(x-2)\vec{i}+(y-3)\vec{j}+(z-4)\vec{k})\}\cdot(\vec{i}+\vec{j}+\vec{k})=0$$

i.e., $$x-2+y-3+z-4=0$$

i.e., $$x+y+z=9$$

which is the required equation.

Example 9.36

Find the equation satisfied by the coordinates of a variable point $P(x, y, z)$ lying in the plane in which the points $A(1, 0, 2)$, $B(2, 2, -1)$, $C(1, 1, 0)$ lie.

Solution

Since A, B, C lie in the plane, the vectors $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CA}$ also lie in the plane. So the vector product of any of these two vectors is a vector orthogonal to the plane. Hence, a vector $\vec{n}$ orthogonal to the plane is given by

$$\vec{n} = \overrightarrow{AC} \times \overrightarrow{BC}.$$

Now,

$$\overrightarrow{AC} = \vec{j} - 2\vec{k} \text{ and } \overrightarrow{BC} = -\vec{i} - \vec{j} + \vec{k}.$$

Therefore,

$$\vec{n} = \overrightarrow{AC} \times \overrightarrow{BC} = -\vec{i} + 2\vec{j} + \vec{k}.$$

Since $\overrightarrow{PC}$ lies in the plane,

$$\vec{n} \perp \overrightarrow{PC} \text{ and so } \vec{n} \cdot \overrightarrow{PC} = 0.$$

Now,

$$\overrightarrow{PC} = (1 - x)\vec{i} + (1 - y)\vec{j} + (0 - z)\vec{k}.$$

Thus, we have

$$(1 - x)(-1) + (1 - y)(+2) + (0 - z)(1) = 0$$

i.e.,

$$x - 2y - z + 1 = 0$$

which is the required equation.

EXERCISE 9.5

1. Prove that the normal to the plane containing the three points whose position vectors are $\vec{a}$, $\vec{b}$, $\vec{c}$ lies in the direction

$$\vec{b} \times \vec{c} + \vec{c} \times \vec{a} + \vec{a} \times \vec{b}.$$

2. If $\vec{p}$, $\vec{q}$, $\vec{r}$ denote the vectors $\vec{b} \times \vec{c}$, $\vec{c} \times \vec{a}$, $\vec{a} \times \vec{b}$ respectively, show that $\vec{a}$ is parallel to $\vec{q} \times \vec{r}$, $\vec{b}$ to $\vec{r} \times \vec{p}$, $\vec{c}$ to $\vec{p} \times \vec{q}$ respectively.

3. If $\vec{a}\cdot\vec{b}\times\vec{c}\neq 0$ and

$$\vec{a'}=\frac{\vec{b}\times\vec{c}}{\vec{a}\cdot\vec{b}\times\vec{c}},\quad \vec{b'}=\frac{\vec{c}\times\vec{a}}{\vec{a}\cdot\vec{b}\times\vec{c}},\quad \vec{c'}=\frac{\vec{a}\times\vec{b}}{\vec{a}\cdot\vec{b}\times\vec{c}}$$

show that

(i) $\vec{a'}\ \ \vec{a}=\vec{b'}\cdot\vec{b}=\vec{c'}\cdot\vec{c}=1$

(ii) $\vec{a'}\cdot\vec{b}=\vec{a'}\cdot\vec{c}=0;\ \vec{b'}\cdot\vec{a}=\vec{b'}\cdot\vec{c}=\vec{0}\,;\vec{c'}\cdot\vec{a}=\vec{c'}\cdot\vec{b}=0$

(iii) $\vec{a'}\cdot\vec{b'}\times\vec{c'}=\dfrac{1}{\vec{a}\cdot\vec{b}\times\vec{c}}$

(iv) If $\vec{a},\vec{b},\vec{c}$ are non-coplanar so are $\vec{a'},\vec{b'},\vec{c'}$.

Note

$\vec{a'},\vec{b'},\vec{c'}$ said to constitute a reciprocal system for $\vec{a},\vec{b},\vec{c}$.

MISCELLANEOUS EXERCISE ON CHAPTER 9

1. If $\vec{a}$ is the position vector of a point A in a plane referred to a pair of rectangular axes in the plane and $\vec{a}=a_1\vec{i}+a_2\vec{j}$, show that points on the vector $\overrightarrow{OA}$ have coordinates

$$(a_1t,\ a_2t),\quad 0\leq t\leq 1$$

and the direction of the vector is along the direction of increasing t.

2. Formulate a result analogous to the result in Question 1 above for three-dimensional vectors and prove it.

Note

In this case the points on the vector have coordinates (a_1t, a_2t, a_3t) and a_1, a_2, a_3 are called *direction numbers* or *direction ratios* of the direction of the position vector $\vec{a}$.

$$\frac{a_1}{(a_1^2+a_2^2+a_3^2)^{\frac{1}{2}}},\quad \frac{a_2}{(a_1^2+a_2^2+a_3^2)^{\frac{1}{2}}},\quad \frac{a_3}{(a_1^2+a_2^2+a_3^2)^{\frac{1}{2}}}$$

are called the *direction cosines* of the vector or the directed line segment $\overrightarrow{OA}$. If the directed line segment is

$$\overrightarrow{P_0P_1},\quad P_0=(x_0,y_0,z_0),\quad P_1=(x_1,y_1,z_1)$$

then points on P_0P_1 have coordinates

$$(x_0\,(1-t)+x_1t,\ y_0\,(1-t)+y_1t,\ z_0\,(1-t)+z_1t)$$

and the direction is that of increasing t, $0 \le t \le 1$. We know that $\overrightarrow{P_0P_1}$ is equivalent to $\overrightarrow{OP_3}$ when P_3 has coordinates $(x_1 - x_0, y_1 - y_0, z_1 - z_0)$. $\overrightarrow{P_0P_1}$ has these as direction ratios, the direction cosines being given by these direction ratios. If l, m, n are the direction cosines of $\overrightarrow{P_0P_1}$, any point (x, y, z) on the line is such that

$$\frac{x - x_0}{l} = \frac{y - y_0}{m} = \frac{z - z_0}{n} \Bigg]$$

3. If A is (a_1, a_2, a_3), $\overrightarrow{AB}$ is vector $\vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$, find the coordinates of B.

4. Find the points dividing the line segments PQ below in the ratio (i) $\dfrac{1}{3}$, (ii) $\dfrac{2}{3}$ of the distance from the initial point P to the terminal point Q:

   (a) $P = (3, 1, 7)$, $Q = (-2, 5, 3)$

   (b) $P = (0, 1, 1)$, $Q = (1, 0, 1)$

5. If A, B are two points in space and if they become respectively A', B' on a translation, show that $\overrightarrow{AB} = \overrightarrow{A'B'}$.

6. If P_1, P_2, P_3, P_4 are points in a plane or space and O the origin of vectors, show that P_4 coincides with O if and only if

$$\overrightarrow{OP_1} + \overrightarrow{P_1P_2} + \overrightarrow{P_2P_3} + \overrightarrow{P_3P_4} = \vec{0}.$$

7. Let $\vec{e_1}, \vec{e_2}, \vec{e_3}$ be the position vectors of the points $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ and $\vec{a}$ be not any one of $\vec{e_1}, \vec{e_2}, \vec{e_3}$. Show that $\vec{a}, \vec{e_1}, \vec{e_2}, \vec{e_3}$ are linearly dependent.

8. Does there exist scalars u, v, w such that

$$u\vec{e}_1 + v\vec{e}_2 + w\vec{e}_3 = \vec{i}$$

where

$$\vec{e}_1 = \vec{k}, \vec{e}_2 = \vec{j} + \vec{k}, \vec{e}_3 = -\vec{j} + 2\vec{k} \,?$$

Why ?

9. Find the unit vector in the direction of

   (i) $3\vec{i} - 6\vec{j} + 2\vec{k}$, (ii) $12\vec{i} - 5\vec{k}$.

10. Prove, using vectors, that the line segment joining the mid-points of the non-parallel sides

of a trapezium is parallel to the remaining sides and is equal to half the sum of their lengths.

11. "The mid-points of two opposite sides of a quadrilateral and the mid-points of the diagonals are the vertices of a parallelogram". Prove using vectors.

12. Show that the projection of $\vec{a}$ on $\vec{b} \neq \vec{0}$ is

$$\left(\frac{\vec{a}\cdot\vec{b}.}{|\vec{b}\,|}\right)\vec{b}.$$

13. Find the projection of the vector $7\vec{i}+\vec{j}-\vec{k}$ on

(a) $2\vec{i}+6\vec{j}+3\vec{k}$, (b) $7\vec{i}+\vec{j}-3\vec{k}$.

14. Find the cosine of the angle between the vectors

$$3\vec{i}-2\vec{j}-6\vec{k} \text{ and } 4\vec{i}-\vec{j}+8\vec{k}.$$

15. Find a vector orthogonal to both

$$\vec{i}+\vec{j}+5\vec{k} \text{ and } 2\vec{i}-\vec{k}.$$

16. Prove that

$$(\vec{a}+\vec{b})^2=(\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b})=|\vec{a}\,|^2+|\vec{b}\,|^2$$

if and only if $\vec{a}, \vec{b}$ are orthogonal.

17. Prove generally that

$$|\vec{a}+\vec{b}\,| \leq |\vec{a}\,|+|\vec{b}\,|.$$

Note : The above inequality is known as triangle inequality.

18. Prove Cauchy - Schwarz inequality

$$(\vec{a}\cdot\vec{b}\,)^2 \leq |\vec{a}\,|^2\,|\vec{b}\,|^2.$$

19. Prove using vectors : If the diagonals of a parallelogram are equal in length, then it is a rectangle.

20. Prove using vectors : The quadrilateral obtained by joining mid-points of adjacent sides of a rectangle is a rhombus.

21. Points L, M, N divide the sides BC, CA, AB of ΔABC in the ratios 1 : 4, 3 : 2, 3 : 7 respectively. Prove that $\overrightarrow{AL}+\overrightarrow{BM}+\overrightarrow{CN}$ is a vector parallel to $\overrightarrow{CK}$, when K divides AB in the ratio 1 : 3.

22. Show that the three points having position vectors

$$\vec{a} - 2\vec{b} + 3\vec{c}, \quad -2\vec{a} + 3\vec{b} + 2\vec{c}, \quad -8\vec{a} + 13\vec{b}$$

are collinear whatever be $\vec{a}, \vec{b}, \vec{c}$.

23. If D, E, F are the mid-points of the sides BC, CA, AB of a triangle show that

$$\overrightarrow{AD} + \overrightarrow{BE} + \overrightarrow{CF} = \vec{0}.$$

24. Show that the four points whose position vectors are

$$6\vec{i} - 7\vec{j}, \quad 16\vec{i} - 19\vec{j} - 4\vec{k}, \quad 3\vec{j} - 6\vec{k}, \quad 2\vec{i} + 5\vec{j} + 10\vec{k}$$

are coplanar.

25. Prove using vectors : The median to the base of an isosceles triangle is perpendicular to the base.

26. Establish using vectors :

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B.$$

27. Find the area of the triangle formed by O, A, B when

$$\overrightarrow{OA} = \vec{i} + 2\vec{j} + 3\vec{k}, \quad \overrightarrow{OB} = -3\vec{i} - 2\vec{j} + \vec{k}.$$

28. Prove that

$$(\vec{a} \times \vec{b}) \times \vec{c} = \vec{a} \times (\vec{b} \times \vec{c})$$

if and only if $(\vec{c} \times \vec{a}) \times \vec{b} = \vec{0}$.

29. Prove using vectors that if a, b, c are the lengths of the sides of a triangle, its area is

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)}$$

where $s = \frac{1}{2}(a + b + c)$.

30. Show that

$$[\vec{b} \times \vec{c}, \vec{c} \times \vec{a}, \vec{a} \times \vec{b}] = [\vec{a}\,\vec{b}\,\vec{c}]^2.$$

31. It is given that

$\vec{x} = \dfrac{\vec{b} \times \vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}]}, \vec{y} = \dfrac{\vec{c} \times \vec{a}}{[\vec{a}\,\vec{b}\,\vec{c}]}$ and $\vec{z} = \dfrac{\vec{a} \times \vec{b}}{[\vec{a}\,\vec{b}\,\vec{c}]}$ where $\vec{a}, \vec{b}, \vec{c}$ are non coplanar vectors.

Find the value of $\vec{x} \cdot (\vec{a} + \vec{b}) + \vec{y} \cdot (\vec{b} + \vec{c}) + \vec{z} \cdot (\vec{c} + \vec{a})$.

32. Find the equation satisfied by the co-ordinates of a variable point $P(x, y, z)$ lying in the plane in which the points $A(1, 0, 2)$, $B(2, 2, -1)$, $C(1, 1, 0)$ lie.

33. If D, E are the mid-points of sides AB and AC of a triangle ABC respectively, show that $\overrightarrow{BE} + \overrightarrow{DC} = \frac{3}{2}\overrightarrow{BC}$.

34. Show that the angle between two diagonals of a cube is $\cos^{-1}\left(\frac{1}{3}\right)$.

35. The position vectors of the foci of an ellipse are $\vec{b}$ and $-\vec{b}$ and the length of the major axis is $2a$. Prove that the equation of the ellipse is $a^4 - a^2(\vec{r}^{\,2} + \vec{b}^{\,2}) + (\vec{b} \cdot \vec{r})^2 = 0$.

CHAPTER 10

Three-dimensional Geometry

10.1 Three-dimensional Space: Geometric Concepts

We have already studied plane geometry or analytic geometry in two dimensions in the earlier class. However, the space we live in is three-dimensional, and, therefore, it is quite natural to extend the scope of analytic geometry to three-dimensions. The necessary preparation has already been made in the last chapter where we studied vectors in three-dimensional space. While doing plane analytic geometry we confine ourselves to coordinate methods. Now that we have learnt a very useful tool of vectors, we shall see that the vector approach makes the study of three-dimensional analytic geometry very simple and elegant. Much though we shall obtain most of the results in vector form using the techniques of vector algebra that we have just learnt, nevertheless we shall also translate these results in the Cartesian form which, at times, presents a more clear geometric and analytic picture of the situation.

We first recall some of the notions learnt in the last chapter pertaining to space geometry.

Three-dimensional Cartesian Coordinate System

We had discussed at length, in Chapter 5 of the textbook for class XI, how to establish a 1 − 1 correspondence between the points in a plane and ordered pairs (x,y) of real numbers through a two-dimensional Cartesian coordinate system, by fixing a point O as origin and two mutually perpendicular lines through O as x-axis and y-axis (say in the plane of the paper). We extend this coordinate system to three-dimensions by taking a third line through O perpendicular to both these axes (perpendicular to the plane of the paper) as the z-axis. The three axes are taken in such a way that they form a *right-handed system*. This means that if a screw, placed at the origin, is turned to move from the positive x-axis to the positive y-axis, it moves up in the direction of the positive z-axis (Fig. 10.1).

The three perpendicular coordinate axes in turn define three *coordinate planes*. The plane passing through the axes Ox and Oy is called the xy-plane (or z plane or xOy plane). Similarly, we have the yz-plane and the zx-plane . It is obvious that these three coordinate planes are also mutually perpendicular.

With any point P we associate the coordinates (x, y, z) in the following way (see §9.1). Drop a perpendicular from P to the xy-plane meeting it at N, take $PN=z$. From N drop perpendiculars on x-axis and y-axis meeting them respectively at A and B. Take $NB=x$ and $NA=y$. Alternatively, from P drop perpendiculars on the axes of x, y and z meeting them respectively at A,B and C. Then $OA=x$, $OB=y$ and $OC=z$. Again , if L, M, N are the feet of the perpendiculars from P on the yOz, zOx, xOy planes, $x=PL$, $y=PM$, $z=PN$. The coordinates x,y and z are taken positive or negative according as the respective points A, B and C are on the positive or negative side of the corresponding coordinate axis. (see note on page 411).

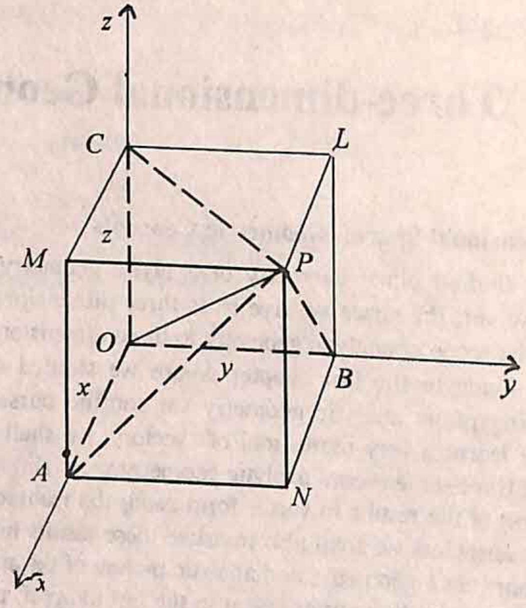

Fig. 10.1

Having associated an ordered triplet of real numbers, (x, y, z) with every point P in space in a unique way, it is easy to see that given a triplet (x, y, z), we can locate a point P in space uniquely whose Cartesian coordinates are (x, y, z). Thus, there exists a 1-1 correspondence between the points in space and the ordered triplets of real numbers (x, y, z).

Base Vectors and Position Vector

The three unit vectors $\vec{i}$, $\vec{j}$ and $\vec{k}$ in the direction of x, y and z -axes respectively are called Cartesian *base vectors.*

If P is any point in space with Cartesian coordinates (x, y, z) and with *position vector* $\vec{r}$ then, as explained in the last chapter (see § 9.2) and as shown in Fig. 10.2, the components of $\vec{r}$ in the directions Ox, Oy and Oz are respectively $x\vec{i}$, $y\vec{j}$, and $z\vec{k}$, and hence

$$\overrightarrow{OP} = \vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

Thus, for any point P (x, y, z), the position vector $\vec{r}$ of P is given by $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$, and conversely the coordinates of any point whose position vector is $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ are (x, y, z).

In other words, having fixed a coordinate system, with every point P in space, we can associate a position vector $\vec{r}$ in a unique manner.

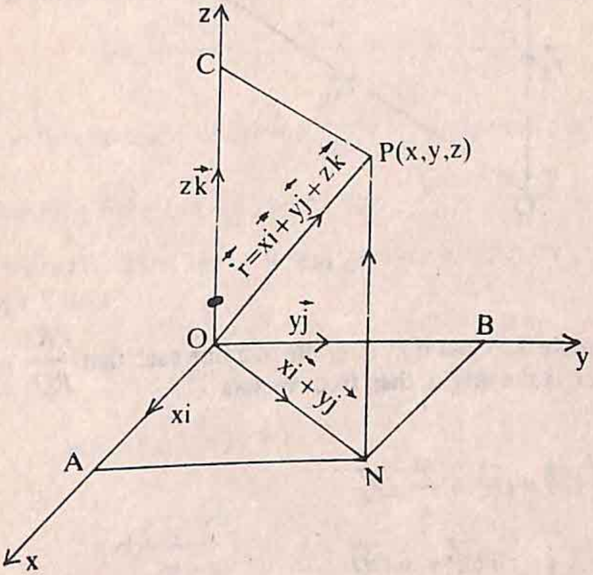

Fig. 10.2

Remark

We must bear in mind that a position vector stands for a typical representative vector in the equivalence class containing vectors equivalent to this position vector.

Distance between Two Points

We have seen in the last chapter that the distance OP of any point $P\ (x, y, z)$ from the origin O is given by

$$OP = |\overrightarrow{OP}| = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$$

Similarly, if $P\ (x_1, y_1, z_1)$ and $Q\ (x_2, y_2, z_2)$ are two points with position vectors $\vec{r_1}$ and $\vec{r_2}$ as shown in Fig 10.3, then

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \vec{r}_2 - \vec{r}_1$$
$$= (x_2 - x_1)\vec{i} + (y_2 - y_1)\vec{j} + (z_2 - z_1)\vec{k}$$

and

$$PQ = |\overrightarrow{PQ}| = |\vec{r_2} - \vec{r_1}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

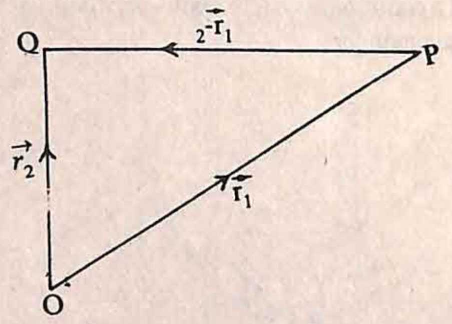

Fig. 10.3

Section Formula

We have seen (see Example 9.5) that if P, R, Q lie in a line such that $\frac{PR}{RQ} = \frac{m}{n}$, and R lies between P and Q , and O is the origin, then from vectors

$$\left(\frac{m}{n}+1\right)\overrightarrow{OR}=\overrightarrow{OP}+\frac{m}{n}\overrightarrow{OQ}$$

i.e.,

$$\overrightarrow{OR}=\frac{n\,\overrightarrow{OP}+m\,\overrightarrow{OQ}}{m+n} \qquad \dots (10.1)$$

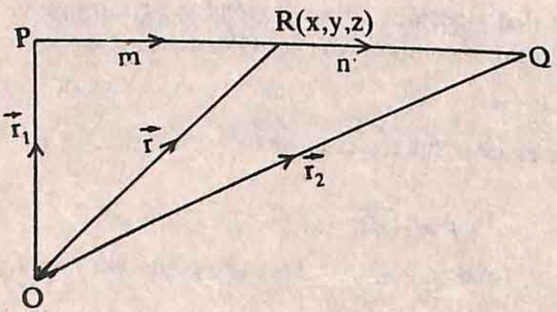

Fig. 10.4

As we have seen already (see Note to Example 9.5), if P is (x_1, y_1, z_1), Q is (x_2, y_2, z_2) and R is (x, y, z), then

$$x=\frac{mx_2+nx_1}{m+n},\ y=\frac{my_2+ny_1}{m+n},\ z=\frac{mz_2+nz_1}{m+n} \qquad \dots (10.2)$$

Notes

1. To get the *mid-point of PQ* (See note to Example 9.5), we put $m = n = 1$ and get

$$x = \frac{x_1 + x_2}{2},\ y = \frac{y_1 + y_2}{2},\ z = \frac{z_1 + z_2}{2}$$

2. If R lies on the extension of PQ and $\frac{PR}{RQ} = \frac{m}{n}$, clearly $m > n$. If $PR = m\,\mu$ and $RQ = n\,\mu$, then $PQ = PR - RQ = (m - n)\,\mu$, so that $\frac{PQ}{QR} = \frac{m-n}{n}$. Thus Q lies between P and R and divides PR in the ratio $(m - n) : n$. Hence, if P is (x_1, y_1, z_1), Q is (x_2, y_2, z_2) and R is (x, y, z), then

$$x_2 = \frac{(m-n)\,x + n\,x_1}{(m-n)+n} = \frac{(m-n)\,x + n\,x_1}{m}$$

i.e. $$mx_2 = (m - n)\,x + nx_1,$$

i.e. $$x = \frac{mx_2 - nx_1}{m - n}$$

Likewise $$y = \frac{my_2 - ny_1}{m - n},\quad z = \frac{mz_2 - nz_1}{m - n}$$

The expressions for x, y, z are the same as for the point R dividing PQ in the ratio $m : n$ *but for changing the sign of* n. This is more often described by saying that in the case of external division in the ratio $m : n$, n is assumed to be negative. Note that, in the case of external division $m > n$ and m is never equal to n..

3. In the case of R lying between P and Q, writing $\frac{PR}{RQ} = \frac{m}{n}$, R has coordinates (x, y, z) given by

$$x = \frac{mx_2 + nx_1}{m + n},\quad y = \frac{my_2 + ny_1}{m + n},\quad z = \frac{mz_2 + nz_1}{m + n}$$

i.e. $x = (1 - \lambda)\,x_1 + \lambda\,x_2$, where $\lambda = \frac{m}{m + n}$

Similarly, $y = (1 - \lambda)\,y_1 + \lambda\,y_2$, $z = (1 - \lambda)\,z_1 + \lambda\,z_2$

Note that P, Q respectively correspond to $m = 0$, $n = 0$ and that $m + n \neq 0$. Thus, points in PQ, inclusive of P and Q correspond to the points with the coordinates

$$\left((1-\lambda)\,x_1+\lambda\,x_2\,,\,(1-\lambda)\,y_1+\lambda\,y_2\,,\ (1-\lambda)\,z_1+z_2\right) \text{ for } 0 \le \lambda \le 1.$$

If R divides PQ externally in the ratio $m : n$, $m > n$, then

$$x=\frac{mx_2-nx_1}{m-n}=1-\frac{m}{m-n}\,x_1+\frac{m}{m-n}\,x_2$$

$$=(1-\lambda)\,x_1+\lambda\,x_2$$

where $\lambda=\dfrac{m}{m-n}\ \ge 1$, $\lambda = 1$ corresponding to Q. We have, similarly

$$y=(1-\lambda)\,y_1+\lambda\,y_2\,;\, z=(1-\lambda)\,z_1+\lambda\,z_2.$$

If, now, R lies to the left of both P and Q, and $\dfrac{PR}{RQ}=\dfrac{m}{n}$, then $m<n$. In this case too, $m=0$ corresponding to $R=P$. If $PR = m\,\mu$, $RQ = n\,\mu$, then

$$\frac{RP}{PQ}=\frac{RP}{RQ-PR}=\frac{m\mu}{n\mu-m\mu}=\frac{m}{n-m}$$

Since P lies between R and Q and divides RQ in the ratio $m : n-m$

$$x_1=\frac{(n-m)\,x+m\,x_2}{n-m+m}=\frac{(n-m)\,x+m\,x_2}{n}$$

i. e., $$x=\frac{nx_1-mx_2}{n-m}=\left(1-\left\{\frac{-m}{n-m}\right\}\right)x_1+\left(\frac{-m}{n-m}\right)x_2$$

$$=(1-\lambda)\,x_1+\lambda\,x_2$$

where $\lambda=\dfrac{-m}{n-m}\ <0$ and $\lambda=0$ corresponds to P.

Likewise

$$y=(1-\lambda)\,y_1+\lambda\,y_2\,,\, z=(1-\lambda)\,z_1+\lambda\,z_2$$

All points to the left of P inclusive of P on the extension of the line segment PQ correspond to points (x, y, z) with $\lambda \le 0$.

Summing up, if P is (x_1, y_1, z_1), Q is (x_2, y_2, z_2), all points on the line joining P and Q have coordinates

$$\left((1-\lambda)\,x_1+\lambda\,x_2,\,(1-\lambda)\,y_1+\lambda\,y_2,\,(1-\lambda)\,z_1+\lambda\,z_2\right)$$

where λ varies through all real numbers. λ is called a parameter in the above representation which is called the parametric form of the line joining P and Q.

Direction Ratios and Direction Cosines

As was pointed out in the note to Exercise 2 of the Miscellaneous Exercises on Chapter 9, if (a, b, c) are the coordinates of a point P, then a, b, c are called the *direction numbers or direction ratios* of $\vec{r} = a\vec{i} + b\vec{j} + c\vec{k}$ or of $\overrightarrow{OP} \neq \vec{0}$. Note that the vector $\lambda\,\overrightarrow{OP}$ where λ is any scalar will have direction ratios $\lambda a, \lambda b, \lambda c$ and that if $\lambda\overrightarrow{OP}$ is opposite in direction to $\overrightarrow{OP}$ all the direction ratios of $\lambda\overrightarrow{OP}$ have signs opposite to that of $\overrightarrow{OP}$. Thus a, b, c or λa, λb, λc for some positive λ can be deemed as direction ratios of $\overrightarrow{OP}$. The angles α, β, γ, which the non-zero vector $\vec{r}$ makes with the base vectors $\vec{i}$, $\vec{j}$ and $\vec{k}$ (on the axis of x, y and z) respectively are called the *direction angles*, and the cosines of these angles, which we denote by

$$l = \cos\alpha, \qquad m = \cos\beta, \qquad n = \cos\gamma$$

are called the *direction cosines* of $\vec{r}$.

Then

$$l = \cos\alpha = \frac{\vec{r}.\vec{i}}{|\vec{r}\,|\,|\vec{i}\,|} = \frac{(a\vec{i} + b\vec{j} + c\vec{k})\cdot\vec{i}}{|\,\vec{r}\,|} = \frac{a}{\sqrt{a^2 + b^2 + c^2}}.$$

Similarly,

$$m = \cos\beta = \frac{b}{\sqrt{a^2 + b^2 + c^2}} \text{ and } n = \cos\gamma = \frac{c}{\sqrt{a^2 + b^2 + c^2}}.$$

It is obvious that the direction cosines satisfy the relation

$$l^2 + m^2 + n^2 = 1.$$

The following points should be noted:

(i) The relation between direction ratios and direction cosines is given by

$$a : b : c = l : m : n \text{ with } l^2 + m^2 + n^2 = 1$$

which should give

$$l = \frac{a}{\sqrt{a^2 + b^2 + c^2}},\ m = \frac{b}{\sqrt{a^2 + b^2 + c^2}},\ n = \frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

denoting as usual the unique positive square root of the number under it.

(ii) Given any vector $\vec{v} = a\vec{i} + b\vec{j} + c\vec{k}$ (not neccessarily a position vector), its direction ratios are a, b, c and its direction cosines are

$$l = \frac{a}{|\vec{v}\,|},\ m = \frac{b}{|\vec{v}\,|},\ n = \frac{c}{|\vec{v}\,|}.$$

(iii) Given two parallel vectors $\vec{v_1}$ and $\vec{v_2}$ $(\vec{v_1}, \vec{v_2} \neq \vec{0})$, i.e. $\vec{v_2} = \lambda \vec{v_1}$, if $\vec{v_1} = a\vec{i} + b\vec{j} + c\vec{k}$, then $\vec{v_2} = \lambda a\vec{i} + \lambda b\vec{j} + \lambda c\vec{k}$, for some real λ; $\vec{v_2}$ has direction ratios λa, λb, λc or a, b, c if $\lambda > 0$. Thus, *vectors having the same direction have equal direction ratios.* If $\lambda < 0$, $\vec{v_2}$ has the direction ratios $-a, -b, -c$.

On the other hand the direction cosines of $\vec{v_2}$ are $\dfrac{\lambda a}{|\lambda \vec{v_1}|}, \dfrac{\lambda b}{|\lambda \vec{v_1}|}, \dfrac{\lambda c}{|\lambda \vec{v_1}|}$ which are the same as the direction cosines of $\vec{v_1}$ only if λ is positive. If λ is negative, the direction cosines of $\vec{v_2}$ are equal in magnitude but opposite in sign to those of $\vec{v_1}$. Hence, *like vectors have equal direction cosines whereas unlike vectors have direction cosines equal in magnitude but opposite in signs.*

(iv) Consider the line joining the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ whose position vectors are $\vec{r_1}$ and $\vec{r_2}$. Then

$$\vec{AB} = \vec{r_2} - \vec{r_1} = (x_2 - x_1)\vec{i} + (y_2 - y_1)\vec{j} + (z_2 - z_1)\vec{k}$$

Then, the direction ratios of $\vec{AB}$ are $(x_2 - x_1) : (y_2 - y_1) : (z_2 - z_1)$ and the direction cosines of $\vec{AB}$ are given by

$$l = \frac{x_2 - x_1}{|\vec{AB}|}, m = \frac{y_2 - y_1}{|\vec{AB}|}, n = \frac{z_2 - z_1}{|\vec{AB}|}.$$

(v) Given any two vectors $\vec{v_1} = a_1\vec{i} + b_1\vec{j} + c_1\vec{k}$ and $\vec{v_2} = a_2\vec{i} + b_2\vec{j} + c_2\vec{k}$, we know that the angle θ between them is given by

$$\cos\theta = \frac{\vec{v_1} \cdot \vec{v_2}}{|\vec{v_1}|\,|\vec{v_2}|} = \frac{a_1a_2 + b_1b_2 + c_1c_2}{|\vec{v_1}|\,|\vec{v_2}|}.$$

If l_1, m_1, n_1 and l_2, m_2, n_2 are the direction cosines of these two vectors, then we can write

$$\cos\theta = l_1l_2 + m_1m_2 + n_1n_2.$$

$$\text{since} \quad l_i = \frac{a_i}{|\vec{v_i}|}, m_i = \frac{b_i}{|\vec{v_i}|}, n_i = \frac{c_i}{|\vec{v_i}|}, i = 1, 2$$

Hence, the two vectors $\vec{v_1}$ and $\vec{v_2}$ are orthogonal if and only if

$$l_1l_2 + m_1m_2 + n_1n_2 = 0.$$

(vi) We know that the unit vector in the direction of $\vec{v}$ is $\dfrac{\vec{v}}{|\vec{v}|}$. This can be written in the alternative form $l\vec{i} + m\vec{j} + n\vec{k}$, where l, m, n are the direction cosines of the vector $\vec{v}$. Hence

$$\vec{v} = |\vec{v}|(l\vec{i} + m\vec{j} + n\vec{k}).$$

(vii) A vector having direction ratios a, b, c is given by $\vec{v} = a\,\vec{i} + b\,\vec{j} + c\,\vec{k}$.

Example 10.1

Show that the three points $A\,(2, 3, 4)$, $B\,(-1, 2, -3)$ and $C\,(-4, 1, -10)$ are collinear and find the ratio in which C divides AB.

Solution

From what you have learnt till now, there are several ways of showing the collinearity of A, B, C.

(a) $$|\overrightarrow{AB}| = \sqrt{(-1-2)^2 + (2-3)^2 + (-3-4)^2} = \sqrt{59}$$

$$|\overrightarrow{BC}| = \sqrt{(-4+1)^2 + (1-2)^2 + (-10+3)^2} = \sqrt{59}$$

$$|\overrightarrow{AC}| = \sqrt{(-4-2)^2 + (1-3)^2 + (-10-4)^2} = 2\sqrt{59}$$

i.e. $AB + BC = AC$ i.e. A, B, C are collinear.

(b) Direction ratios of $\overrightarrow{AB}$ are

$$(-1-2), (2-3), (-3-4), \text{ i.e. } -3, -1, -7.$$

Direction ratios of $\overrightarrow{BC}$ are

$$(-4+1), (1-2), (-10+3), \text{ i.e. } -3, -1, -7.$$

Hence, $\overrightarrow{AB} \parallel \overrightarrow{BC}$ i.e. A, B, C are collinear.

(c) Any point on the line AB, dividing the segment in the ratio $\lambda : 1$ is

$$\left(\frac{-\lambda+2}{\lambda+1}, \frac{2\lambda+3}{\lambda+1}, \frac{-3\lambda+4}{\lambda+1}\right).$$

For this to be the point C, we must have

$$\frac{-\lambda+2}{\lambda+1} = -4, \frac{2\lambda+3}{\lambda+1} = 1, \frac{-3\lambda+4}{\lambda+1} = -10.$$

The first equation gives $\lambda = -2$ and this satisfies the second and third equation also. Hence, C lies on the line joining A and B.

This gives the answer to the second part of the problem also, viz., that C, divides AB in the ratio $-2 : 1$, which is the same as $2 : -1$ or $2 : 1$ externally.

Example 10.2

Find the position vector of a point P such that $\overrightarrow{OP}$ is inclined to Ox at 45° and to Oy at 60° and $OP = 12$ units.

Solution

Given that

$$l = \cos 45^\circ = \frac{1}{\sqrt{2}}\ ,\ m = \cos 60^\circ = \frac{1}{2},\ n = \cos\gamma$$

$l^2 + m^2 + n^2 = 1$ gives $n^2 = \dfrac{1}{4}$ or $n = \pm\ \dfrac{1}{2} = \cos\gamma$. Hence, $\overrightarrow{OP}$ is inclined to OZ either at 60° or at 120°. Now

$$\vec{r} = |\vec{r}|\,(l\,\vec{i} + m\,\vec{j} + n\,\vec{k}) = 12\left(\frac{1}{\sqrt{2}}\vec{i} + \frac{1}{2}\vec{j} \pm \frac{1}{2}\vec{k}\right)$$

i.e.

$$\vec{r} = 6\sqrt{2}\,\vec{i} + 6\,\vec{j} \pm 6\,\vec{k}.$$

EXERCISE 10.1

1. Find the coordinates of the point which divides the join of the points (2, −1, 3) and (4, 3, 1) in the ratio 3 : 4 internally.
2. A line passes through the points (6, −7, −1) and (2, −3, 1). Find direction ratios and the direction cosines of the line so directed that the angle α made with Ox is acute.
3. A vector $\vec{r}$ has length 21 and direction ratios (2, −3, 6). Find the direction cosines and components of $\vec{r}$.
4. The direction ratios of a line are 1, −3, 2. Find its direction cosines.
5. Find the direction cosines of the vectors
   - (a) $2\vec{i} + 2\vec{j} - \vec{k}$
   - (b) $6\vec{i} - 2\vec{j} - 3\vec{k}$
   - (c) $3\vec{i} + 4\vec{k}$
6. Find the angle at which the following vectors are inclined to the positive direction of each of the coordinate axes:
   - (a) $\vec{i} - \vec{j} + \vec{k}$
   - (b) $4\,\vec{i} + 8\,\vec{j} + \vec{k}$
   - (c) $\vec{j} - \vec{k}$
7. A vector $\vec{v}$ is inclined at equal angles to Ox, Oy and Oz. If magnitude of $\vec{v}$ is 6 units, find $\vec{v}$.

8. If A and B are two points whose position vectors are $3\vec{i}+\vec{j}-2\vec{k}$ and $\vec{i}-3\vec{j}-\vec{k}$ respectively. Find the position vectors of the points dividing AB

   (a) internally in the ratio 1 : 3

   (b) externally in the ratio 3 : 1

9. Find the foot of the perpendicular drawn from the point A (1, 0, 3) to the join of the points B (4, 7, 1) and C (3, 5, 3).

10. A (1, 0, 4), B (0, –11, 3), C (2, –3, 1) are three points and D is the foot of the perpendicular from A on BC. Find the coordinates of D.

11. Calculate the cosine of the angle A of the triangle with vertices A (1, –1, 2), B (6, 11, 2), C (1, 2, 6).

10.2 Straight Line in Space

We know that a straight line is uniquely determined in space if

(a) it passes through a given point and has a given direction, or

(b) it passes through two given points.

We shall now determine the vector equation as well as Cartesian equations of a straight line under the above conditions.

(a) *Equation of a Straight Line passing through a fixed point A and Collinear to a given vector* $\vec{m}$.

Let $\vec{a}$ be the position vector of the fixed point A, and $\vec{r}$, the position vector of any arbitrary point P on the line as indicated in Fig. 10.5.

Then, P is a point on the line if and only if $\overrightarrow{AP}$ is parallel to the vector $\vec{m}$, i.e. $\overrightarrow{AP}=\lambda\vec{m}$, where λ is some real number (positive, negative or zero).

But $$\overrightarrow{AP}=\overrightarrow{OP}-\overrightarrow{OA} \quad \text{i.e.} \quad \lambda\vec{m}=\vec{r}-\vec{a}.$$

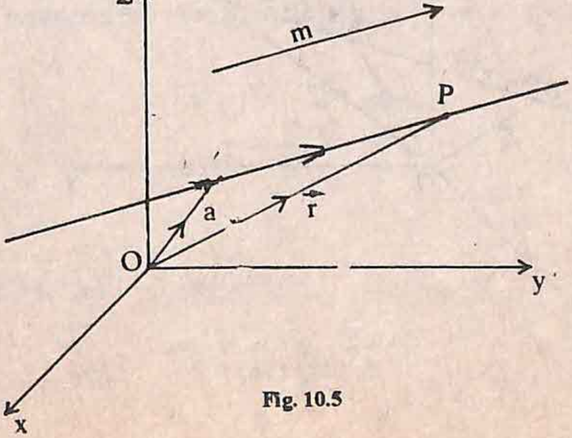

Fig. 10.5

Conversely, for each value of (the parameter) λ, this equation gives the position vector of a point P on the line. Hence, the vector equation of the line is

$$\vec{r} = \vec{a} + \lambda \vec{m} \qquad \text{... (10.3)}$$

Cartesian Form

Let the coordinates of the fixed point A be (x_1, y_1, z_1) and the direction ratios of the line be a, b, c. Then

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}, \vec{a} = x_1\vec{i} + y_1\vec{j} + z_1\vec{k} \text{ and } \vec{m} = a\vec{i} + b\vec{j} + c\vec{k}.$$

Substituting these values in (10.3) and comparing the coefficients of $\vec{i}, \vec{j}$ and $\vec{k}$, we get the following *parametric equation* of the line :

$$x = x_1 + \lambda a, y = y_1 + \lambda b, z = z_1 + \lambda c \qquad \text{... (10.4)}$$

Eliminating the parameter λ from (10.4), we get the following *Cartesian equations* of the line:

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} \qquad \text{... (10.5)}$$

We thus see that the equation of a straight line passing through a given point and parallel to a given direction can be expressed in one of the above three forms viz., the vector form (10.3), the parametric form (10.4), or the Cartesian form (10.5). (x_1, y_1, z_1) can be any one point on the given straight line. In view of (10.5), the equations are independent of the choice of (x_1, y_1, z_1).

(b) *Equation of a Straight Line passing through Two Fixed Points*

Now consider a line in space passing through two fixed distinct points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ whose position vectors are $\vec{a}$ and $\vec{b}$ respectively, as shown in Fig. 10.6.

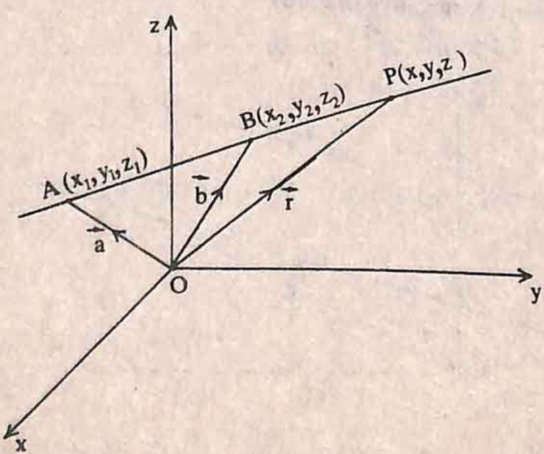

Fig. 10.6

If $\vec{r}$ is the position vector, $\overrightarrow{OP}$, of an arbitrary point $P(x, y, z)$, then P is a point on the line $\Leftrightarrow \overrightarrow{AP} = \lambda \overrightarrow{AB}$ for some real λ. Now

$$\overrightarrow{AB} = \vec{b} - \vec{a} \quad \text{and} \quad \overrightarrow{AP} = \vec{r} - \vec{a}$$

Therefore, P is on the line if and only if

$$\vec{r} - \vec{a} = \lambda(\vec{b} - \vec{a})$$

i.e.,

$$\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a}) \qquad \text{... (10.6)}$$

Cartesian Form

Using the Cartesian components of $\vec{a}$ and $\vec{b}$, the above equation becomes

$$x\vec{i} + y\vec{j} + z\vec{k} = x_1\vec{i} + y_1\vec{j} + z_1\vec{k}$$
$$+ \lambda\,[(x_2 - x_1)\,\vec{i} + (y_2 - y_1)\,\vec{j} + (z_2 - z_1)\,\vec{k}\,]$$

Hence, eliminating the parameter λ, the Cartesian equations of a line passing through two points (x_1, y_1, z_1) and (x_2, y_2, z_2) become

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1} \qquad \text{...(10.7)}$$

Angle between Two Lines

Let

$$\vec{r} = \vec{a} + \lambda\vec{b} \qquad \text{... (10.8)}$$

$$\vec{r} = \vec{a} + \mu\vec{b'} \qquad \text{...(10.9)}$$

and

be two straight lines in space. These straight lines are in the directions of $\vec{b}$ and $\vec{b'}$ respectively. The angle θ between these lines is defined as the angle between the directions of $\vec{b}$ and $\vec{b'}$. We observe that the angle between a pair of straight lines depends only on their directions and not on their positions. As

$$\vec{b} \cdot \vec{b'} = |\vec{b}\,|\,|\vec{b'}\,| \cos\theta,$$

$$\therefore \qquad \cos\theta = \frac{\vec{b} \cdot \vec{b'}}{|\vec{b}\,|\,|\vec{b'}|} \qquad \text{... (10.10)}$$

Cartesian Form

If the equations of the lines are in Cartesian form:

$$\frac{x - x_1}{b_1} = \frac{y - y_1}{b_2} = \frac{z - z_1}{b_3},$$

and

$$\frac{x - x'_1}{b'_1} = \frac{y - y'_1}{b'_2} = \frac{z - z'_1}{b'_3},$$

then the angle between them would be given by

$$\cos \theta = \frac{b_1 b'_1 + b_2 b'_2 + b_3 b'_3}{\sqrt{b_1^2 + b_2^2 + b_3^2}\ \sqrt{b_1'^2 + b_2'^2 + b_3'^2}} \qquad \text{...(10.11)}$$

If instead the direction ratios we take the direction cosines, the formula becomes

$$\cos \theta = ll' + mm' + nn' \qquad \text{... (10.12)}$$

Note that the condition for orthogonality of the two lines is

$$b_1 b'_1 + b_2 b'_2 + b_3 b'_3 = 0 \qquad \text{... (10.13)}$$

or,

$$ll' + mm' + nn' = 0 \qquad \text{... (10.14)}$$

Example 10.3

Find the equation of the line (both vector and Cartesian) which is parallel to the vector $2\vec{i} - \vec{j} + 3\vec{k}$ and which passes through the point (5, –2, 4).

Solution

The vector equation of the line is

$$\vec{r} = (5\vec{i} - 2\vec{j} + 4\vec{k}\ ) + \lambda\,(2\vec{i} - \vec{j} + 3\vec{k})$$

Now $\vec{r}$ is the position vector of any point $P\,(x, y, z)$ on the line, i.e.

$$x\vec{i} + y\vec{j} + z\vec{k} = 5\vec{i} - 2\vec{j} + 4\vec{k} + \lambda(2\vec{i} - \vec{j} + 3\vec{k}\,)$$

$$= (5 + 2\lambda)\vec{i} + (-2 - \lambda)\vec{j} + (4 + 3\lambda)\vec{k}$$

Thus,

$$x = 5 + 2\lambda$$

$$y = -2 - \lambda$$

$$z = 4 + 3\lambda$$

This is the parametric form of the equation. Eliminating λ, we get

$$\frac{x - 5}{2} = \frac{y + 2}{-1} = \frac{z - 4}{3},$$

which is the equation of the line in Cartesian form.

Remark

Note that the direction ratios of a line in the direction of $2\vec{i} - \vec{j} + 3\vec{k}$ are 2, −1, 3 so that the above Cartesian equations are immediate.

Shortest Distance between Two Lines

By the shortest distance between two lines is meant the join of a point in one with one on the other so that the length of the segment so obtained is the smallest.

If two lines in space intersect at a point, then obviously the shortest distance between them is zero. Lines which do not intersect and are also not parallel are called *skew lines*. For skew lines, the direction of the shortest distance would be perpendicular to both the lines.

The shortest distance is then determined in the following way: Take any point S on one line and another point T on the other. Then the magnitude of the shortest distance vector would be equal to that of the projection of $\overrightarrow{ST}$ along the direction of the line of shortest distance.

Consider two skew lines l_1 and l_2(Fig. 10.7), whose directions are those of $\vec{b}_1$ and $\vec{b}_2$ and whose equations are

$$\vec{r}_1 = \vec{a}_1 + \lambda \vec{b}_1$$

and

$$\vec{r}_2 = \vec{a}_2 + \lambda \vec{b}_2.$$

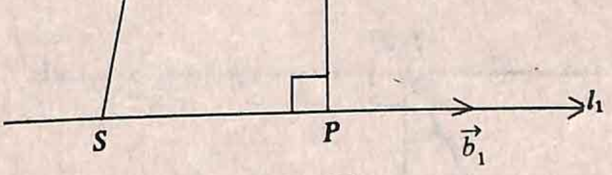

Fig. 10.7

If $\overrightarrow{PQ}$ is the shortest distance vector between them, then being perpendicular to both $\vec{b}_1$ and $\vec{b}_2$ it must be parallel to $\vec{b}_1 \times \vec{b}_2$. The unit vector $\vec{n}$ along $\overrightarrow{PQ}$ would therefore be

$$\vec{n} = \frac{\vec{b}_1 \times \vec{b}_2}{|\vec{b}_1 \times \vec{b}_2|}$$

Let

$$\overrightarrow{PQ} = d\,\vec{n}$$

where d is the magnitude of the shortest distance vector. Consider the line segment ST, where S is the point on l_1 with position vector a_1 and T on l_2 with position vector $\vec{a_2}$. If θ is the angle between $\vec{ST}$ and $\vec{PQ}$, then, as explained above,

$$PQ = ST\cos\theta$$

But

$$\cos\theta = \frac{\vec{PQ}\cdot\vec{ST}}{|\vec{PQ}||\vec{ST}|} = \frac{d\,\vec{n}\cdot(\vec{a}_2-\vec{a}_1)}{d\,ST} = \frac{(\vec{b}_1\times\vec{b}_2)\cdot(\vec{a}_2-\vec{a}_1)}{ST\,|\vec{b}_1\times\vec{b}_2|}$$

Hence,

$$d = PQ = ST\,|\cos\theta| = \frac{(\vec{b}_1\times\vec{b}_2)\cdot(\vec{a}_2-\vec{a}_1)}{|\vec{b}_1\times\vec{b}_2|} \qquad \ldots (10.15)$$

Notes

(1) The two lines will intersect if and only if $d = 0$ i.e. if and only if

$$(\vec{b}_1\times\vec{b}_2)\cdot(\vec{a}_2-\vec{a}_1) = 0.$$

(2) If two lines l_1, l_2 are parallel and given respectively by

$$\vec{r}_1 = \vec{a}_1 + \lambda\vec{b}, \quad \vec{r}_2 = \vec{a}_2 + \mu\vec{b},$$

then obviously they are coplanar. In this case the (shortest) distance between them is given by

$$d = \left|\frac{\vec{b}\times(\vec{a}_2-\vec{a}_1)}{|\vec{b}|}\right|,$$

since $d = PQ = ST\cos(90-\theta)$ (See Fig. 10.8) so that

$$d = ST\,|\sin\theta| = |\vec{a}_2-\vec{a}_1|\left|\frac{\vec{b}\times(\vec{a}_2-\vec{a}_1)}{|\vec{b}|\,|\vec{a}_2-\vec{a}_1|}\right|$$

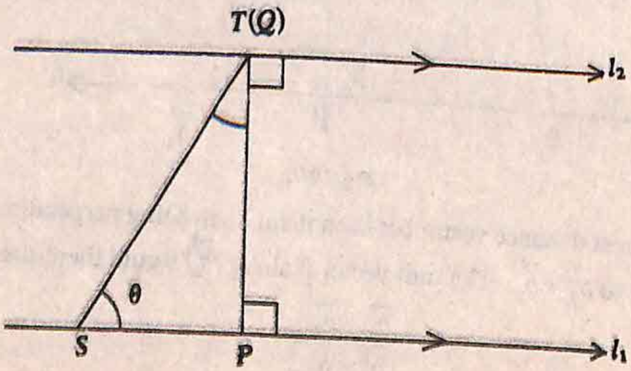

Fig. 10.8

(3) The reader is required to write down the formulas for shortest distance in Cartesian form.

EXERCISE 10.2

1. The Cartesian equations of a line are

$$\frac{x-5}{3} = \frac{y+4}{7} = \frac{z-6}{2}$$

Find a vector equation for the line.

2. A line passes through the point with position vector $2\vec{i} - \vec{j} + 4\vec{k}$ and is in the direction of $\vec{i} + \vec{j} - 2\vec{k}$. Find equations for the line in vector and in Cartesian form.

3. Find the vector equation for the line through the points $A\,(3, 4-7)$ and $B\,(1, -1, 6)$.

4. Find the coordinates of the point where the line through $A\,(3, 4, 1)$ and $B\,(5, 1, 6)$ crosses the xy-plane.

5. The points $A\,(4, 5, 10)$, $B\,(2, 3, 4)$ and $C\,(1, 2, -1)$ are three vertices of a parallelogram $ABCD$. Find vector and Cartesian equations for the sides AB and BC and find the coordinates of D.

6. The Cartesian equations of a line are $6x - 2 = 3y + 1 = 2z - 2$. Find the direction ratios of the line and write down the vector equation of the line through $(2, -1, -1)$ which is parallel to the given line.

7. Find the angle between each of the following pair of lines:

(a) $\vec{r}_1 = 3\vec{i} + 2\vec{j} - 4\vec{k} + \lambda\,(\vec{i} + 2\vec{j} + 2\vec{k})$
$\vec{r}_2 = 5\vec{j} - 2\vec{k} + \mu\,(3\vec{i} + 2\vec{j} + 6\vec{k})$

(b) A line with direction ratios 2 : 2 : 1.
A line joining (3, 1, 4) to (7, 2, 12).

(c) $\dfrac{x+4}{3} = \dfrac{y-1}{5} = \dfrac{z+3}{4}$

$\dfrac{x+1}{1} = \dfrac{y-4}{1} = \dfrac{z-5}{2}$

8. Find the angle between the following pairs of lines:

(a) $\dfrac{x-2}{3} = \dfrac{y+1}{-2}, z = 2$

$\dfrac{x-1}{1} = \dfrac{2y+3}{3} = \dfrac{z+5}{2}$

(b) $\vec{r}_1 = 4\vec{i} - \vec{j} + \lambda\,(\vec{i} + 2\vec{j} - 2\vec{k})$
$\vec{r}_2 = \vec{i} - \vec{j} + 2\vec{k} - \mu\,(2\vec{i} + 4\vec{j} - 4\vec{k})$

9. Find the angle between the pair of lines with direction ratios

   (a) $5, -12, 13\ ;\ -3, 4, 5$

   (b) $1, 1, 2\ ; \sqrt{3}-1, -\sqrt{3}-1, 4$

10. Find the shortest distance between the lines whose vector equations are

$$\vec{r}_1 = \vec{i} + \vec{j} + \lambda(2\vec{i} - \vec{j} + \vec{k})$$

and $$\vec{r}_2 = 2\vec{i} + \vec{j} - \vec{k} + \mu(3\vec{i} - 5\vec{j} + 2\vec{k})$$

11. The vector equations of two lines are

$$\vec{r}_1 = \vec{i} + 2\vec{j} + \vec{k} + \lambda(\vec{i} - \vec{j} + \vec{k})$$
$$\vec{r}_2 = 2\vec{i} - \vec{j} - \vec{k} + \mu(2\vec{i} + \vec{j} + 2\vec{k}).$$

Find the shortest distance between these lines.

12. The vector equations of two lines are

$$\vec{r}_1 = (1-t)\vec{i} + (t-2)\vec{j} + (3-2t)\vec{k}$$
$$\vec{r}_2 = (s+1)\vec{i} + (2s-1)\vec{j} - (2s+1)\vec{k}.$$

Find the shortest distance between these lines.

13. Determine whether the following pair of lines intersect :

   (a) $\vec{r}_1 = \vec{i} - \vec{j} + \lambda(2\vec{i} + \vec{k})\ ;\ \ \vec{r}_2 = 2\vec{i} - \vec{j} + \mu(\vec{i} + \vec{j} - \vec{k})$

   (b) $\vec{r}_1 = \vec{i} + \vec{j} - \vec{k} + \lambda(3\vec{i} - \vec{j})\ ;\ \vec{r}_2 = 4\vec{i} - \vec{k} + \mu(2\vec{i} + 3\vec{k})$

   (c) $\dfrac{x-1}{2} = \dfrac{y+1}{3} = z;\quad \dfrac{x+1}{5} = \dfrac{y-2}{1};\quad z = 2.$

10.3 Plane

A particular plane can be specified in several ways, for example,

(a) One and only one plane can be drawn through three non-collinear points; therefore, three given points specify a particular plane.

(b) One and only one plane can be drawn to contain two concurrent lines; therefore, two given concurrent lines specify a particular plane.

(c) One and only one plane can be drawn perpendicular to a given direction at a given distance from the origin; therefore, the normal to a plane and the distance of the plane from the origin specify a particular plane.

(d) One and only one plane can be drawn through a given point and perpendicular to a given direction; therefore, a point on the plane and a normal to the plane specify a particular plane.

There are many other ways of specifying a particular plane but the most useful are described in (c) and (d) as these lead to a very simple form for the vector equation of a plane.

Consider a plane which is at a distance d from the origin and which is perpendicular to the unit vector $\vec{n}$ as shown in Fig. 10.9.

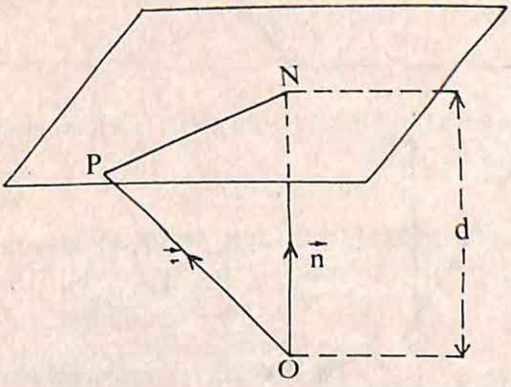

Fig. 10.9

If ON is the perpendicular from the origin to the plane then

$$\overrightarrow{ON} = d\vec{n}$$

If P is any point on the plane, $\overrightarrow{NP}$ is perpendicular to $\overrightarrow{ON}$, and if P is not on the plane, $\overrightarrow{NP}$ is not perpendicular to $\overrightarrow{ON}$. Therefore,

$$P \text{ is a point on the plane} \Leftrightarrow \overrightarrow{NP} \cdot \overrightarrow{ON} = 0 \quad \dots (10.15)$$

This equation is called the scalar product form of the vector equation of the plane.

If $\vec{r}$ is the position vector of P, $\overrightarrow{NP} = \vec{r} - d\,\vec{n}$. Therefore (10.15) becomes

$$(\vec{r} - d\,\vec{n}) \cdot d\vec{n} = 0$$

$$\Rightarrow \quad (\vec{r} - d\,\vec{n}) \cdot \vec{n} = 0$$

$$\Rightarrow \quad \vec{r} \cdot \vec{n} - d\,\vec{n} \cdot \vec{n} = 0$$

$$\Rightarrow \quad \vec{r} \cdot \vec{n} = d \quad \dots (10.16)$$

as $\vec{n} \cdot \vec{n} = 1$.

The equation (10.16) is the standard form of the vector equation of the plane, where $\vec{r}$ is the position vector of any point on the plane, $\vec{n}$ is the unit vector perpendicular to the plane and d is the distance of the plane from the origin.

Cartesian Equation of a Plane

Let

$$\vec{r} \cdot \vec{p} = d \quad \dots (10.17)$$

be the vector equation of a plane, where $\vec{p}$ is the unit vector normal to the plane as shown in Fig. 10.10.

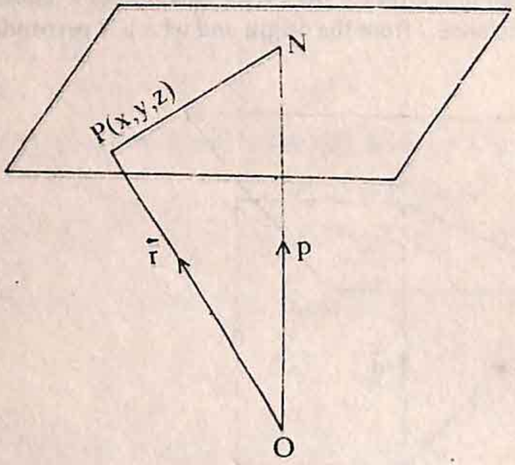

Fig. 10.10

If $P(x, y, z)$ is any point on the plane, then

$$\overrightarrow{OP} = \vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

and if l, m, n are the direction cosines of $\vec{p}$ then

$$\vec{p} = l\vec{i} + m\vec{j} + n\vec{k}.$$

Substituting in (10.17), we get

$$(x\vec{i} + y\vec{j} + z\vec{k}) \cdot (l\vec{i} + m\vec{j} + n\vec{k}) = d$$

$$\Rightarrow \quad lx + my + nz = d \qquad \text{... (10.18)}$$

which is the Cartesian equation of the plane. Here l, m, n are the direction cosines of the normal to the plane and d is the length of the perpendicular from the origin to the plane. If α, β, γ are the angles made by $\vec{p}$ with Ox, Oy, Oz, then (10.18) takes the form

$$x\cos\alpha + y\cos\beta + z\cos\gamma = d.$$

Compare this with the corresponding two dimensional form

$$x\cos\alpha + y\sin\alpha \equiv x\cos\alpha + y\cos\left(\frac{\pi}{2} - \alpha\right) = p$$

Remark

(10.18) shows that if $\vec{r} \cdot (A\vec{i} + B\vec{j} + C\vec{k}) = D$ is the vector equation of a plane, then $Ax + By + Cz = D$ is the Cartesian equation of the plane, where A, B, C are the direction ratios of the normal to the plane. So there is an easy transformation between these two forms for the equation of the plane. Since (10.17) always represents a plane, it is clear that any equation of the form $Ax + By + Cz = D$ will always represent a plane.

Plane passing through a given Point and Perpendicular to a given Direction

From Example 9.34, the vector equation of the plane passing through $\vec{a}$ and having $\vec{n}$

as the normal from origin is

$$(\vec{r}-\vec{a})\cdot\vec{n}=0$$

If the components of $\vec{a}$ are (a_1, a_2, a_3) and the direction ratios of $\vec{n}$ are n_1, n_2, n_3, then the Cartesian form of the equation becomes

$$(x-a_1)n_1+(y-a_2)n_2+(z-a_3)n_3=0$$

Note

n_1, n_2, n_3 are called the *normal direction ratios* or *cosines* (as the case may be) of the plane.

Example 10.4

Find the direction cosines of the perpendicular from the origin to the plane

$$\vec{r}\cdot(6\vec{i}-3\vec{j}-2\vec{k})+1=0$$

Solution

We write the given equation as

$$\vec{r}\cdot(6\vec{i}-3\vec{j}-2\vec{k})=-1$$

Now, $$|6\vec{i}-3\vec{j}-2\vec{k}|=\sqrt{36+9+4}=7.$$

So that $\frac{6}{7}\vec{i}-\frac{3}{7}\vec{j}-\frac{2}{7}\vec{k}$ is the unit vector in the direction of $6\vec{i}-3\vec{j}-2\vec{k}$. Then

$$\vec{r}\cdot\left(-\frac{6}{7}\vec{i}+\frac{3}{7}\vec{j}+\frac{2}{7}\vec{k}\right)=\frac{1}{7}$$

is the equation of the plane in the form $\vec{r}\cdot\vec{n}=d$ of (10.17). This shows that perpendicular vector from the origin to the plane is

$$\vec{n}=-\frac{6}{7}\vec{i}+\frac{3}{7}\vec{j}+\frac{2}{7}\vec{k}.$$

Thus, the direction cosines of $\vec{n}$ are $-\frac{6}{7}, \frac{3}{7}, \frac{2}{7}$.

Example 10.5

Find the equation of the plane through the points $P\,(2, 2, -1)$, $Q\,(3, 4, 2)$, $R\,(7, 0, 6)$.

Solution

The general equation of a plane through $(2, 2, -1)$ is

$$a(x-2)+b(y-2)+c(z+1)=0 \qquad \text{... (10.19)}$$

It passes through Q and R if

$$a + 2b + 3c = 0$$

and
$$5a - 2b + 7c = 0.$$

These give

$$\frac{a}{20} = \frac{b}{8} = \frac{c}{-12} \text{ or } \frac{a}{5} = \frac{b}{2} = \frac{c}{-3} = k, \text{ say}$$

Substituting in (10.19), we see that the required equation is

$$5x + 2y - 3z - 17 = 0.$$

Angle between Two Planes

The angle between two planes P_1, P_2 is defined as the angle between the normals to P_1, P_2 from the origin O (which can be chosen arbitrarily). It is the angle between the lines in which P_1, P_2 intersect a plane perpendicular to both of them.

Suppose P_1 and P_2 are two planes specified by the vector equations

$$\vec{r} \cdot \vec{n_1} = d_1$$
$$\vec{r} \cdot \vec{n_2} = d_2$$

as shown in Fig. 10.11. Note that the plane determined by $\vec{n_1}$ and $\vec{n_2}$ is perpendicular to both P_1 and P_2.

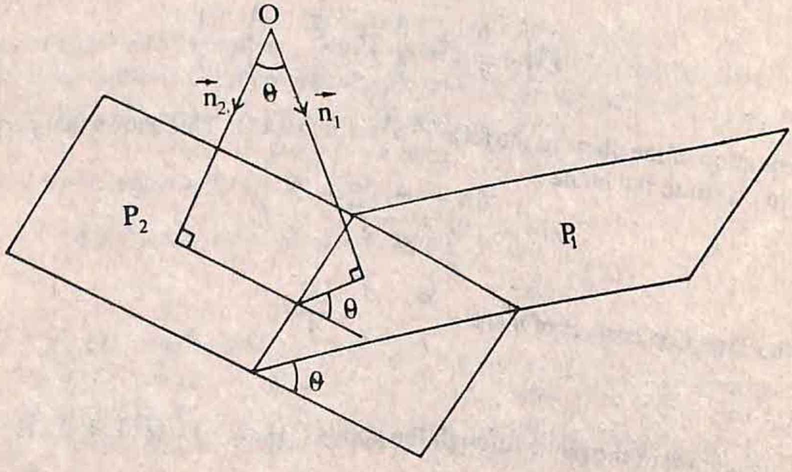

Fig. 10.11

Therefore, if θ is the angle between P_1 and P_2

$$\cos \theta = \vec{n_1} \cdot \vec{n_2} \qquad \text{... (10.20)}$$

Remark

The planes are perpendicular to each other if $\vec{n_1} \cdot \vec{n_2} = 0$ and parallel if $\vec{n_1} = \pm \vec{n_2}$.

Example 10.6

Find the angle between the planes whose vector equations are

$$\vec{r} \cdot (\vec{i} + \vec{j} - 2\vec{k}) = 3 \quad \text{and} \quad \vec{r} \cdot (2\vec{i} - 2\vec{j} + \vec{k}) = 2.$$

Solution

If θ is the angle between the two planes,

$$\cos\theta = \frac{(\vec{i} + \vec{j} - 2\vec{k})}{\sqrt{6}} \cdot \frac{(2\vec{i} - 2\vec{j} + \vec{k})}{3} = -\frac{\sqrt{6}}{9}.$$

This is the cosine of the obtuse angle between the planes. The acute angle is $\cos^{-1}\left(\frac{\sqrt{6}}{9}\right)$.

Example 10.7

Find the equation of the plane through the points (2, 2, 1) and (9, 3, 6) and perpendicular to the plane

$$2x + 6y + 6z - 1 = 0.$$

Solution

Any plane through (2, 2, 1) is

$$a(x-2) + b(y-2) + c(z-1) = 0 \qquad \text{... (10.21)}$$

It passes through (9, 3, 6) if

$$a(9-2) + b(3-2) + c(6-1) = 0$$

i.e.,

$$7a + b + 5c = 0 \qquad \text{... (10.22)}$$

The plane (10.21) is perpendicular to the given plane if

$$2a + 6b + 6c = 0 \qquad \text{... (10.23)}$$

From (10.22) and (10.23), we have

$$\frac{a}{-24} = \frac{b}{-32} = \frac{c}{40} \quad \text{i.e.,} \quad \frac{a}{3} = \frac{b}{4} = \frac{c}{-5}$$

Substituting in (10.21), the required equation of the plane is

$$3x + 4y - 5z = 9.$$

Angle between a Line and a Plane

Consider the line

$$\vec{r} = \vec{a} + \lambda\vec{b}$$

and the plane

$$\vec{r}\cdot\vec{n} = d.$$

The angle between the line *l* and the plane *P* is defined to be the angle between *l* and the line of intersection of the planes containing *l* and perpendicular to *P*, i.e., it is equal to the complement of the angle between the normal to the plane and the given line.

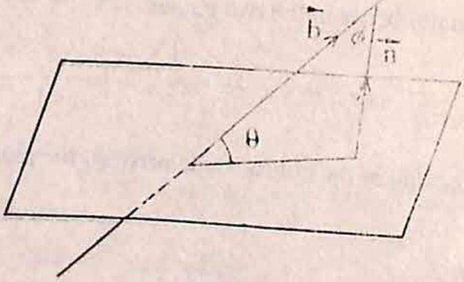

Fig. 10.12

The angle between the line and a normal to the plane is given by

$$\cos\phi = \frac{\vec{b}\cdot\vec{n}}{|\vec{b}|} \quad \text{... (10.24)}$$

If θ is the angle between the line and the plane, then $\theta = \frac{\pi}{2} - \phi$

i.e., $$\sin\theta = \cos\phi \quad \text{... (10.25)}$$

Therefore, the angle between the line $\vec{r} = \vec{a} + \lambda\vec{b}$ and the plane $\vec{r}\cdot\vec{n} = d$ is given by

$$\sin\theta = \frac{\vec{b}\cdot\vec{n}}{|\vec{b}|} \quad \text{... (10.26)}$$

Example 10.8

Find the angle between the line

$$\vec{r} = (\vec{i} + 2\vec{j} - \vec{k}) + \lambda(\vec{i} - \vec{j} + \vec{k})$$

and the plane

$$\vec{r}\cdot(2\vec{i} - \vec{j} + \vec{k}) = 4.$$

Solution

Here, $$\sin\theta = \frac{(\vec{i}-\vec{j}+\vec{k})}{\sqrt{3}}\cdot\frac{(2\vec{i}-\vec{j}+\vec{k})}{\sqrt{6}} = \frac{2\sqrt{2}}{3}$$

Therefore, $$\theta = \sin^{-1}\left(\frac{2\sqrt{2}}{3}\right).$$

Distance of a Point from a Plane

Consider a point P with position vector $\vec{a}$ and a plane π_1 whose equation is

$$\vec{r}\cdot\vec{n} = d.$$

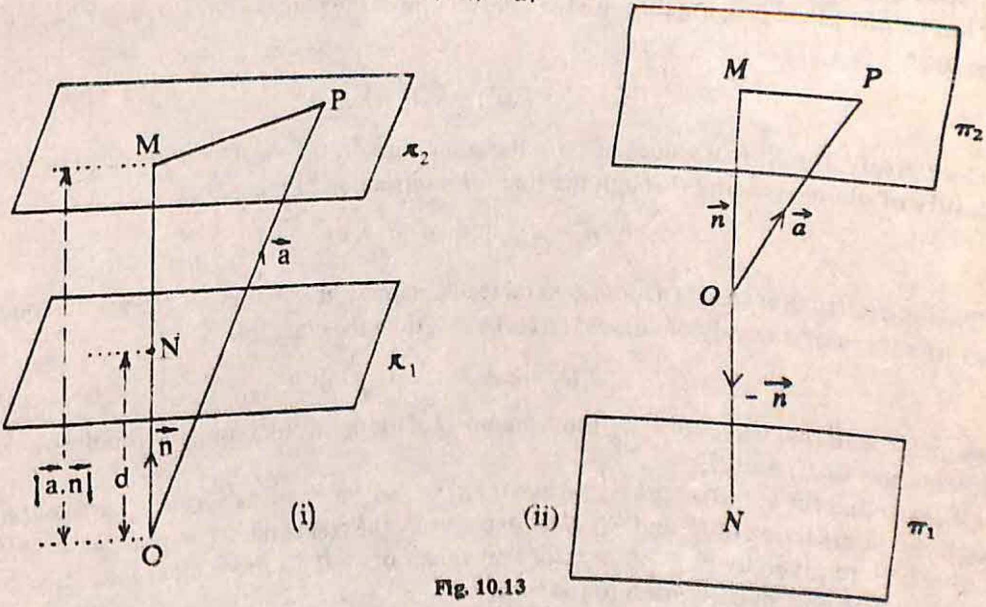

Fig. 10.13

Consider the plane π_2 through P parallel to the plane π_1. The unit normal to π_2 is $\vec{n}$ or $-\vec{n}$ and hence its equation is $(\vec{r}-\vec{a})\cdot\vec{n} = 0$, which can be written as

$$\vec{r}\cdot\vec{n} = \vec{a}\cdot\vec{n}$$

i.e. the distance OM of this plane from the origin is $|\vec{a}\cdot\vec{n}|$. ... (10.27)

Therefore, the distance MN of P from the plane π_1[See Fig. 10.13 (i)] is

$$OM - ON = |\vec{a}\cdot\vec{n}| - d \quad \text{... (10.28)}$$

You may establish the similar result for Fig. 10.13 (ii) yourself.

Family of Planes passing through the Intersection of Given Planes

If π_1 and π_2 are planes with equations $\vec{r}\cdot\vec{n}_1 = d_1$ and $\vec{r}\cdot\vec{n}_2 = d_2$ respectively, the position vector of any point on the line of intersection must satisfy both the equations.

If $\vec{t}$ is the position vector of a point on this line, then

$$\vec{t}\cdot\vec{n_1} = d_1 \quad \text{and} \quad \vec{t}\cdot\vec{n_2} = d_2$$

Therefore, for any value of k_1, k_2

$$k_1\,\vec{t}\cdot\vec{n_1} + k_2\,\vec{t}\cdot\vec{n_2} = k_1d_1 + k_2d_2$$

or,

$$\vec{t}\cdot[k_1\vec{n_1} + k_2\vec{n_2}] = k_1d_1 + k_2d_2 \qquad \dots (10.29)$$

But the equation $\vec{r}\cdot(k_1\vec{n_1} + k_2\vec{n_2}) = k_1d_1 + k_2d_2$ represents a plane π_3 which is such that if any vector $\vec{r}$ statisfies both the equations of π_1 and π_2, it also satisfies the equation of π_3.

That is, any plane passing through the intersection of the planes $\vec{r}\cdot\vec{n_1} = d_1$ and $\vec{r}\cdot\vec{n_2} = d_2$ has an equation

$$\vec{r}\cdot(k_1\vec{n_1} + k_2\vec{n_2}) = k_1d_1 + k_2d_2.$$

Conversely, for all real values of k_1, k_2 the equation $\vec{r}\cdot(k_1\vec{n_1} + k_2\vec{n_2}) = k_1d_1 + k_2d_2$ represents the family of planes passing through the lines of intersection of the planes

$$\vec{r}\cdot\vec{n_1} = d_1 \quad \text{and} \quad \vec{r}\cdot\vec{n_2} = d_2.$$

This is a particular case of a more general result, namely, if $E_1 = 0$ and $E_2 = 0$ are the equations of two members of a family of curves (or surfaces), then the equation

$$k_1E_1 + k_2E_2 = 0 \qquad \dots (10.30)$$

represents, for all real values of k_1,k_2, those members of the family that contain the point (or points) of intersection of E_1 and E_2.

Observe that for $k_1 = 0$ we get $E_2 = 0$ and for $k_2 = 0$ we get $E_1 = 0$ as curves or surfaces passing through the intersection of E_1 and E_2. If we exclude E_2, the rest of these curves or surfaces, as the case may be, are given by $E_1 + \lambda E_2 = 0$ for real values of λ. If a surface other than E_2 is needed the above equation can be written in the form

$$E_1 + kE_2 = 0 \qquad \text{(by making } k = \frac{k_2}{k_1}\text{)}$$

Let

$$\pi_1 : ax + by + cz = d$$

$$\pi_2 : a'x + b'y + c'z = d'$$

be two planes. Putting $x = 0$ in the two equations

$$\left.\begin{aligned} by + cz &= d \\ b'y + c'z &= d' \end{aligned}\right\} \qquad \dots (10.31)$$

Recall that if π_1 and π_2 are parallel, then their normal direction ratios, viz; $a, b, c; a', b', c'$ are such that

$$\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$$

Thus, if they are *not parallel*, at least one of the pairs $\left(\frac{a}{a'}, \frac{b}{b'}\right)$, $\left(\frac{b}{b'}, \frac{c}{c'}\right)$ or $\left(\frac{c}{c'}, \frac{a}{a'}\right)$ is different from the other two. Assume that π_1, π_2 are not parallel to each other and, without loss of generality, that $\frac{b}{b'} \neq \frac{c}{c'}$, then the system of linear equations in (10.31) has a unique solution, say, y_0, z_0; more explicitly $y_0 = \frac{c'd - cd'}{bc' - b'c}$, $z_0 = \frac{b'd - bd'}{bc' - b'c}$. This means that, if $x_0 = 0$, then (x_0, y_0, z_0) lies on both π_1, π_2. Note that this is the point common to the yOz-plane and the planes π_1 and π_2.

If l, m, n are direction ratios of a line perpendicular to the normals from origin to π_1 and π_2, then

$$\left.\begin{aligned} al + bm + cn &= 0 \\ a'l + b'm + c'n &= 0 \end{aligned}\right\} \qquad \ldots (10.32)$$

since the direction ratios of π_1, π_2 are respectively a, b, c ; a', b', c'.

Hence,

$$\frac{l}{bc' - b'c} = \frac{m}{ca' - c'a} = \frac{n}{ab' - a'b} = k, \text{ say,}$$

where, if at all, only one of the denominators may be 0 as pointed out earlier. Consider now the point whose coordinates are

$$(x, y, z) = (x_0 + kl, y_0 + km, z_0 + kn)$$

where (x_0, y_0, z_0) is the point lying on both π_1, π_2 found earlier, k being any real number. In view of (10.31) and (10.32), (x, y, z) lies on both π_1, π_2 for every real k. But such an (x, y, z) satisfies the equations

$$\frac{x - x_0}{l} = \frac{y - y_0}{m} = \frac{z - z_0}{n} \qquad (10.33)$$

i.e., (x, y, z) lies on the line with above equations. Thus the entire line having equations (10.33) lies on both the planes π_1 and π_2.

Conversely any point common to π_1, π_2 lies on (10.33). For, if (x, y, z) is such a point, then the line joining (x, y, z) and (x_0, y_0, z_0) lies on both the planes and hence is perpendicular to the normals to both the planes from the origin. The choice of l, m, n satisfying (10.32) therefore ensures that the direction ratios of the line segment joining (x, y, z) and (x_0, y_0, z_0) are proportional in view of (10.33) to the direction ratios of a line perpendicular to the normals to both π_1, π_2. Thus (10.33) gives the equations satisfied by all points common to π_1, π_2. Thus, we have shown that

Two planes π_1, π_2, which are not parallel to each other, intersect along a straight line.

Note: The above result can be established using equations in vectors too.

Consider two intersecting planes π_1 and π_2 and let $\vec{a}$ be the position vector of any point common to them. Then, we can write their equations as

$$\pi_1 : (\vec{r} - \vec{a}) \cdot \vec{n_1} = 0$$
$$\pi_2 : (\vec{r} - \vec{a}) \cdot \vec{n_2} = 0$$

where $\vec{n_1}$ and $\vec{n_2}$ are the normals to π_1 and π_2 respectively from the origin.

i.e.,$(\vec{r} - \vec{a})$ is perpendicular to both $\vec{n_1}$ and $\vec{n_2}$, then it is in the direction of $\vec{n_1} \times \vec{n_2}$.

Hence, $\vec{r} - \vec{a} = \vec{k}(\vec{n_1} \times \vec{n_2}) = k\vec{p}$ (say)

i.e., $\vec{r} = \vec{a} + k\vec{p}$

which is the vector equation of a straight line.

In Cartesian system, therefore, the two simultaneous equations

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$

would represent a straight line, provided the normal direction ratios $a_1, b_1, c_1; a_2, b_2, c_2$ are not proportional.

Example 10.9

Find the equation of the plane passing through the intersection of the planes

$$x + y + z = 6 \quad \text{and} \quad 2x + 3y + 4z + 5 = 0$$

and the point (1, 1, 1).

Solution

The plane

$$x + y + z - 6 + k(2x + 3y + 4z + 5) = 0 \qquad \text{... (10.34)}$$

passes through the line of intersection of the given planes for all values of k. It will pass through (1, 1, 1) if

$$-3 + 14k = 0, \text{ i.e. } k = \frac{3}{14}$$

Putting $k = \dfrac{3}{14}$ in (10.34), we get

$$20x + 23y + 26z - 69 = 0$$

which is the required equation of the plane.

EXERCISE 10.3

1. Show that the line L whose vector equation is

$$\vec{r} = 2\vec{i} - 2\vec{j} + 3\vec{k} + \lambda(\vec{i} - \vec{j} + 4\vec{k})$$

is parallel to the plane π whose vector equation is

$$\vec{r} \cdot (\vec{i} + 5\vec{j} + \vec{k}) = 5$$

and find the distance between them.

2. Show that the plane whose vector equation is

$$\vec{r} \cdot (\vec{i} + 2\vec{j} - \vec{k}) = 3$$

contains the line whose vector equation is

$$\vec{r} = \vec{i} + \vec{j} + \lambda(2\vec{i} + \vec{j} + 4\vec{k}).$$

3. Find the vector equation of the line passing through the point (3, 1, 2) and perpendicular to the plane $\vec{r} \cdot (2\vec{i} - \vec{j} + \vec{k}) = 4$. Find also the point of intersection of this line and the plane.

4. Find the vector equation of the following planes in scalar product form:
   (a) $\vec{r} = \vec{i} - \vec{j} + \lambda(\vec{i} + \vec{j} + \vec{k}) + \mu(\vec{i} - 2\vec{j} + 3\vec{k})$
   (b) $\vec{r} = 2\vec{i} - \vec{k} + \lambda\vec{i} + \mu(\vec{i} - 2\vec{j} - \vec{k})$
   (c) $\vec{r} = (1 + s - t)\vec{i} + (2 - s)\vec{j} + (3 - 2s + 2t)\vec{k}$

5. Find the Cartesian equation of the following planes:
   (a) $\vec{r} \cdot (\vec{i} + \vec{j} - \vec{k}) = 2$
   (b) $\vec{r} \cdot (2\vec{i} + 3\vec{j} - 4\vec{k}) = 1$
   (c) $\vec{r} = (s - 2t)\vec{i} + (3 - t)\vec{j} + (2s + t)\vec{k}$

6. Find the vector equation in scalar product form of the plane that contains the lines

$$\vec{r} = (\vec{i} + \vec{j}) + s(\vec{i} + 2\vec{j} - \vec{k})$$

and

$$\vec{r} = (\vec{i} + \vec{j}) + t(-\vec{i} + \vec{j} - 2\vec{k}).$$

7. Find the vector equation of the line through the origin which is perpendicular to the plane

$$\vec{r} \cdot (\vec{i} - 2\vec{j} + \vec{k}) = 3.$$

8. Find a normal vector to the plane

$$x + 2y + 3z - 6 = 0.$$

9. Find the angles between the planes

$$2x - 3y + 4z = 1 \quad \text{and} \quad -x + y = 4.$$

10. Find the equation of the plane through $P\,(1, 4, -2)$ that is parallel to the plane

$$-2x + y - 3z = 0.$$

11. Find the distance from $P\,(2, 1, -1)$ to the plane

$$x - 2y + 4z = 9$$

12. Find the equations of the bisector planes of the angles between the planes

$$3x - 2y + 6z + 8 = 0 \quad \text{and} \quad 2x - y + 2z + 3 = 0.$$

13. Find the equation of the plane containing the line $\dfrac{x+1}{-3} = \dfrac{y-3}{2} = \dfrac{z+2}{1}$ and the point $(0, 7, -7)$.

14. Show that the sum of the reciprocals of the squares of intercepts on rectangular axes made by a fixed plane is same for all systems of rectangular axes with a given origin.

15. Prove that volume of a tetrahedron with three concurrent edges a, b, c is $\dfrac{1}{6}\,[a\ b\ c]$.

16. Establish by vector methods, the equation of the plane making intercepts a, b, c, on the co-ordinates axes in the form

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$$

10.4 Sphere

A sphere is the set of all points in space which are at a constant distance from a fixed point. The constant distance is called the radius and the fixed point is called the centre of the sphere. Let $\vec{r}$ be the position vector of any point on the sphere. If the radius of the sphere is a and the position vector of the centre is $\vec{c}$, then the vector equation of the sphere is (see Fig. 10.14)

$$|\vec{r} - \vec{c}\,| = a \qquad \text{... (10.35)}$$

or,

$$(\vec{r} - \vec{c}\,) \cdot (\vec{r} - \vec{c}\,) = a^2 \qquad \text{... (10.36)}$$

In Cartesian coordinates this takes the form

$$(x - c_1)^2 + (y - c_2)^2 + (z - c_3)^2 = a^2$$

i.e.,

$$x^2 + y^2 + z^2 - 2c_1 x - 2c_2 y - 2c_3 z + c_1^2 + c_2^2 + c_3^2 - a^2 = 0 \qquad \text{... (10.37)}$$

Note that in the above equation only x^2, y^2, z^2, x, y, z and constant terms occur on the left hand side. Further, the centre of the sphere is (c_1, c_2, c_3) and its radius is a. We can write it in the form

$x^2 + y^2 + z^2 - 2fx - 2gy - 2hz + c = 0$, where $c_1 = f$, $c_2 = g$, $c_3 = h$ and $c_1^2 + c_2^2 + c_3^2 - a^2 = c$.

In other words, we can say that if $x^2 + y^2 + z^2 - 2fx - 2gy - 2hz + c = 0$, then (x, y, z) lies on a sphere centred at (f, g, h) and radius $\sqrt{g^2 + f^2 + h^2 - c}$

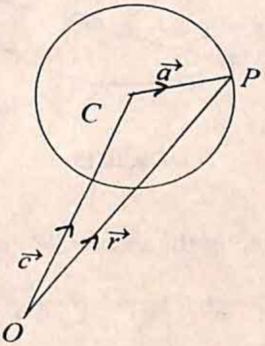

Fig. 10.14

In particular, the equation of a sphere whose centre is origin and radius is a is

$$|\vec{r}| = a \qquad \text{... (10.38)}$$

This corresponds to the Cartesian equation

$$x^2 + y^2 + z^2 = a^2 \qquad \text{... (10.39)}$$

The section or, more precisely the intersection of a sphere by a plane (with a plane) is either void, a singleton or a circle.

Let the centre of the sphere be O and its radius be a units. If the plane π is such that the perpendicular distance of π from O is p, and if $p > a$, any point on the plane π is at a distance greater than or equal to p and so strictly greater than a. Hence no point of π can be on the sphere since any point on the sphere is at a distance equal to a from O. If, now, $p = a$ the foot of the perpendicular from O to π is the only point on π which is at a distance $p = a$ from O. All other points on π are at a distance greater than a from O and hence this intersection is a singleton.

Finally, if $0 \le p < a$, then the distance of any point common to the plane and the sphere from the foot L of the perpendicular from O to π is constantly $\sqrt{a^2 - p^2}$ (See Fig. 10.15). So these points lie on a circle with centre L and radius $\sqrt{a^2 - p^2}$. Thus the assertion made above is completely established.

Note: If π passes through the centre O of the sphere, the radius of the circular section is equal to the radius a of the sphere.

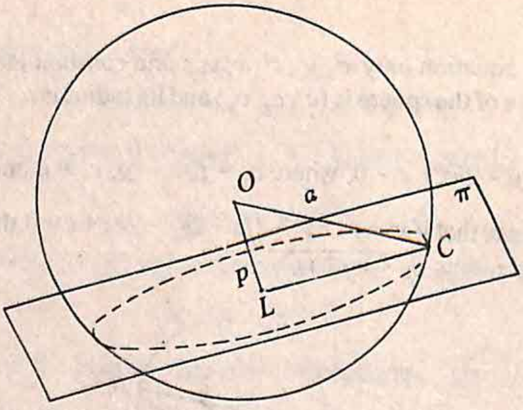

Fig. 10.15

Example 10.10

Find the radius of the circular section of the sphere $|\vec{r}| = 5$ by the plane

$$\vec{r} \cdot (\vec{i} + \vec{j} + \vec{k}) = 3\sqrt{3}$$

Solution

The sphere $|\vec{r}| = 5$ has centre at the origin and radius 5.

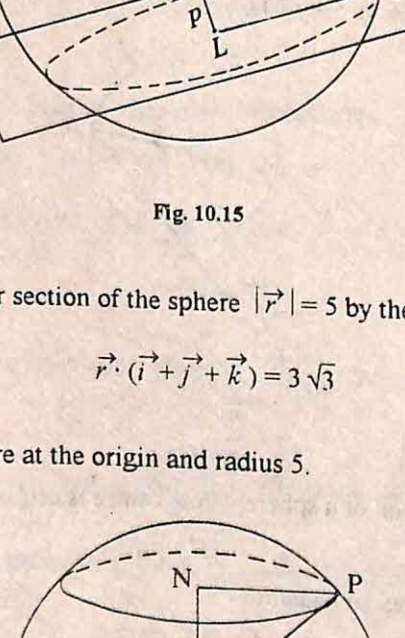

Fig. 10.16

The plane $\vec{r} \cdot (\vec{i} + \vec{j} + \vec{k}) = 3\sqrt{3}$ can be rewritten as $\vec{r} \cdot \frac{1}{\sqrt{3}} (\vec{i} + \vec{j} + \vec{k}) = 3$ in which $\frac{1}{\sqrt{3}} (\vec{i} + \vec{j} + \vec{k})$ is a unit vector.

This shows that the distance of the plane from the origin is 3. Thus, in Fig. 10.16

$$OP = 5\,, \quad ON = 3.$$

Hence $$NP^2 = OP^2 - ON^2 = 5^2 - 3^2 = 4^2$$

i.e., $$NP = 4.$$

Equation of a Sphere, the Extremities of the Diameter being Given

Let AB be the diameter of a sphere whose centre is O (Fig. 10.17). Let the position vectors of the extremities A and B of the diameter be $\vec{a}$ and $\vec{b}$ respectively. Consider a point P on the surface of the sphere with position vector $\vec{r}$. Then APB determines a plane in which O lies. So the section of the sphere by this plane is a circle centred at O. We know that the angle APB is a right angle (angle in a semicircle). In other words, the vectors $\overrightarrow{AP}$ and $\overrightarrow{BP}$ are orthogonal.

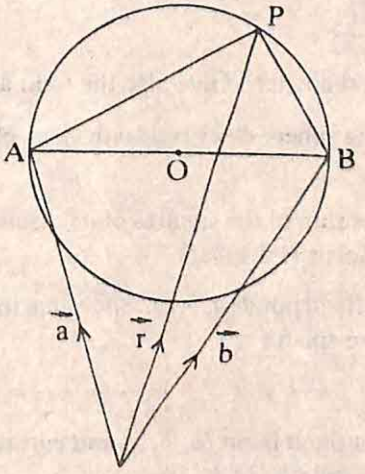

Fig. 10.17

But $$\overrightarrow{AP} = \vec{r} - \vec{a}$$

and $$\overrightarrow{BP} = \vec{r} - \vec{b}$$

The orthogonality of $\overrightarrow{AP}$ and $\overrightarrow{BP}$ implies that

$$(\vec{r} - \vec{a}) \cdot (\vec{r} - \vec{b}) = 0 \qquad \text{... (10.40)}$$

which is the required equation of the sphere. If $a = (a_1, a_2, a_3)$, $b = (b_1, b_2, b_3)$ in Cartesian coordinates, the equation of the sphere on the line segment joining a and b as diameter is

$$(x - a_1)(x - b_1) + (y - a_2)(y - b_2) + (z - a_3)(z - b_3) = 0$$

EXERCISE 10.4

1. Find the equation of the sphere whose centre is $(3, -1, 2)$ and radius is 4.
2. Find the equation of the sphere whose centre is $(1, 1, 1)$ and radius is 1.
3. Find the centre and the radius of the sphere whose equation is
$$x^2 + y^2 + z^2 - 6x + 8y - 10z + 1 = 0.$$
4. Find the centres and the radii of the following spheres:
   (a) $2x^2 + 2y^2 + 2z^2 - 2x + 4y + 2z + 3 = 0$
   (b) $x^2 + y^2 + z^2 - ax - by - cz = 0$
5. Find the equation of the sphere described on the join of the points
   (a) $A\ (3, 4, 5)$, $B\ (5, 2, 3)$
   (b) $A\ (0, 1, 0)$, $B\ (3, -5, 2)$
   as the opposite ends of a diameter. Give also the radii and the centres of the spheres.
6. Obtain the equation of the sphere described on the join of the points $A\ (2, -3, 4)$, $B\ (-5, 6, -7)$ as a diameter.
7. A point moves so that the sum of the squares of its distances from the six faces of a cube is constant. Show that its locus is a sphere
8. A plane passes through a fixed point (a, b, c). Show that the locus of the foot of perpendicular to it from the origin is the sphere
$$x^2 + y^2 + z^2 - ax - by - cz = 0$$
9. A plane passes through a fixed point (a, b, c) and cuts the axes in A, B, C. Show that the locus of the centre of the sphere $OABC$ is
$$\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 2$$
10. Find the equation of the sphere on the join of
    (a) $(2, -3, 1)$ and $(1, -2, -1)$ as diameter.
    (b) $(2, -3, 4)$ and $(-5, 6, -7)$ as diameter.

ANSWERS

EXERCISE 6.1

1. (a) $\int \sec x \tan x \, dx = \sec x$ (b) $\int -\operatorname{cosec} x \cot x \, dx = \operatorname{cosec} x$

(c) $\int \frac{1}{1+x^2} \, dx = \tan^{-1} x$ (d) $\int 2 \cos 2x \, dx = \sin 2x$

(e) $\int 2x \, e^{x^2} \, dx = e^{x^2}$

2. (a) $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$ (b) $\frac{d}{dx}\left(\frac{1}{2} e^{2x}\right) = e^{2x}$

(c) $\frac{d}{dx}\left(-\frac{1}{3} \operatorname{cosec}(3x+4)\right) = \operatorname{cosec}(3x+4)\cot(3x+4)$

(d) $\frac{d}{dx}(-2\cos x) = 2\sin x$

3. (a) $\frac{x^4}{4} + c$ (b) $-\frac{x^{-4}}{4} + c$ (c) $\frac{x^4}{4} - \frac{x^{-4}}{4} + c$

(d) $x^4 + x^{-4} + c$ (e) $\log\sqrt{x} + c$ (f) $\frac{x^2}{2} + \log x + c$

(g) $\frac{x^2}{2} + \frac{1}{x} + c$ (h) $\frac{a}{3} x^3 + \frac{b}{2} x^2 + cx + d$

4. (a) $\sec x + c$ (b) $-\cos x + \sin x + c$

(c) $\tan x + \sec x + c$ (d) $\frac{3}{5} x^{\frac{5}{3}} + c$

5. (a) $-\frac{\cos 2x}{2} + c$ (b) $\frac{e^{2x}}{2} + c$ (c) $\frac{4}{3} e^{3x} + x + c$

5. (d) $\dfrac{(ax+b)^3}{3a}+c$

EXERCISE 6.2

1. $\log \sin x + c$

2. $-\cos(x^2+1)+c$

3. $\dfrac{(\log x)^2}{2}+c$

4. $\dfrac{(\log x^2)^2}{4}+c$

5. $\dfrac{(\log x)^3}{3}+c$

6. $\cos(\cos x)+c$

7. $\dfrac{1}{2a}\sin^2(ax+b)+c$

8. $\dfrac{2a}{3}(ax+b)^{\frac{3}{2}}+c$

9. $\dfrac{4}{3}(x^2+x+1)^{\frac{3}{2}}+c$

10. $\dfrac{(3x-4)^4}{12}+c$

11. $\dfrac{2}{5}(x+2)^{\frac{5}{2}}-\dfrac{4}{3}(x+2)^{\frac{3}{2}}+c$

12. $-\dfrac{\tan(7-4x)}{4}+c$

13. $-\log(1+\cos x)+c$

14. $\dfrac{1}{1+\cos x}+c$

15. $\dfrac{(\log \sin x)^2}{2}+c$

16. $-\cos(\tan^{-1}x)+c$

17. $\log|\sec x|+c$

18. $-\dfrac{3}{2}x-\dfrac{13}{4}\log|3-2x|+c$

19. $\dfrac{1}{2}\{\log(e^{2x}-2)\}+c$

20. $\log|\operatorname{cosec} x-\cot x|+c$

21. $\dfrac{1}{2}\{x-\log|\sin x+\cos x|\}+c$

22. $\dfrac{1}{2}\{x-\log|\cos x-\sin x|\}+c$

23. $-\tan^{-1}(\cos x)+c$

24. $\dfrac{1}{2}\log|6\cos x+4\sin x|+c$

EXERCISE 6.3

1. $\frac{1}{2}\cos x - \frac{1}{14}\cos 7x + c$

2. $\frac{1}{2}x - \frac{\sin(4x + 10)}{8} + c$

3. $\frac{x}{4} + \frac{1}{16}\sin 4x + \frac{\cos 8x}{32} + \frac{\sin 12x}{48} + c$

4. $\frac{\sin 3x}{12} + \frac{3}{4}\sin x + c$

5. $\frac{\cos(6x+3)}{24} - \frac{3}{8}\cos(2x + 1) + c$

6. $\frac{\cos 6x}{192} - \frac{3}{64}\cos 2x + c$

7. $\frac{\cos 6x}{24} - \frac{1}{8}\cos 2x - \frac{1}{6}\cos 4x + c$

8. $\frac{\sin 2x}{4} - \frac{\sin 6x}{12} + c$

9. $2\tan\frac{x}{2} - x + c$

10. $x - \tan\frac{x}{2} + c$

11. $\frac{3x}{8} + \frac{\sin 4x}{32} - \frac{\sin 2x}{4} + c$

12. $\frac{3}{8}x + \frac{\sin 4x}{8} + \frac{\sin 8x}{64} + c$

13. $x - \sin x + c$

14. $\frac{1}{2}\left\{\frac{\sin(m+n)x}{m+n} - \frac{\sin(m-n)x}{m-n}\right\} + c$

15. $-\frac{1}{2}\left\{\frac{\cos(m+n)x}{m+n} + \frac{\cos(m-n)x}{m-n}\right\} + c$

16. $\frac{1}{2}x - \frac{\sin 2bx}{4b} + c$

17. $-\cot x - x + c$

18. $4\tan x - 9\cot x - 25x + c$

19. $-2\cot 2x + c$

20. $\tan x - \sec x + c$

21. $-\cot x - \operatorname{cosec} x + c$

22. $\log|\sin x| + \frac{\sin^8 x}{8} - 2\sin^2 x + \frac{3}{2}\sin^4 x - \frac{2}{3}\sin^6 x + c$

23. $\frac{2\cos^3 x}{3} - \frac{\cos^5 x}{5} - \cos x + c$

EXERCISE 6.4

1. $\log\ x|+\sqrt{x^2+4}|+c$

2. $\frac{1}{2}\log|2x+\sqrt{4x^2+1}|+c$

3. $-\log|(2-x)+\sqrt{x^2-4x+5}|+c$

4. $-\log|(2-x)+\sqrt{x^2-4x+3}|+c$

5. $2\tan^{-1}\frac{x}{2}+c$

6. $\tan^{-1}(x+2)+c$

7. $\frac{1}{6}\log\left|\frac{1+3x}{1-3x}\right|+c$

8. $\frac{1}{6}\log\left|\frac{3x-1}{3x+1}\right|+c$

9. $\frac{1}{5}\sin^{-1}\frac{5x}{3}+c$

10. $\frac{1}{b}\log|bx+\sqrt{a^2+b^2x^2}|+c$

11. $\frac{1}{b}\sin^{-1}\frac{bx}{a}+c$

12. $-\frac{1}{2ab}\log\left|\frac{a-bx}{a+bx}\right|+c$

13. $\frac{3}{2\sqrt{2}}\tan^{-1}(\sqrt{2}x^2)+c$

14. $\frac{1}{6}\log\left|\frac{1+x^3}{1-x^3}\right|+c$

15. $\frac{1}{3}\log|x^3+\sqrt{4+x^6}|+c$

16. $\sqrt{x^2+1}-\log|x+\sqrt{x^2+1}|+c$

17. $\sqrt{x^2-1}-\log|x+\sqrt{x^2-1}|+c$

18. $\frac{a}{4}\log|x^4+c^2|+\frac{b}{2c}\tan^{-1}\frac{x^2}{c}+d$

19. $\frac{1}{3}\log|x^3+\sqrt{x^6-a^6}|+c$

20. $\frac{1}{3}\log|x^3+\sqrt{x^6+a^6}|+c$

21. $\frac{1}{\sqrt{6}}\tan^{-1}\frac{x+1}{\sqrt{6}}+c$

22. $\frac{1}{2\sqrt{2}}\tan^{-1}\frac{2x-1}{\sqrt{2}}+c$

23. $-\frac{1}{3(3x+1)}+c$

24. $\frac{1}{\sqrt{3}}\tan^{-1}\frac{x+2}{\sqrt{3}}+c$

25. $\frac{1}{17}\log\left|\frac{3x-2}{x+5}\right|+c$

26. $\log\left|\frac{2x-3}{2}+\sqrt{x^2-3x+2}\right|+c, (x<1 \text{ or } x>2)$

27. $\frac{1}{\sqrt{2}}\sin^{-1}(4x-1)+c$

28. $\sin^{-1}(x-1)+c$

29. $\sin^{-1}\left(\frac{2x-3}{\sqrt{41}}\right)+c$

30. $\log\left|\frac{2x-a-b}{2}+\sqrt{(x-a)(x-b)}\right|+c$

31. $2\sqrt{2x^2+x-3}+c$

32. $\sqrt{x^2-1}+2\log\left|x+\sqrt{x^2-1}\right|+c$

33. $\sin^{-1}\left(\frac{\sin x}{2}\right)+c$

34. $\log\left|\tan x+\sqrt{16+\tan^2 x}\right|+c$

35. $\sqrt{x^2+x+1}-\frac{1}{2}\left[\log\left|\left(\frac{2x+1}{2}\right)+\sqrt{x^2+x+1}\right|\right]+c$

36. $\frac{x\sqrt{1+x^2}}{2}-\frac{1}{2}\log\left|x+\sqrt{1+x^2}\right|+c$

37. $x-\tan^{-1}x+c$

38. $x+\log\left|\frac{(x+2)^3}{x+1}\right|+c$

EXERCISE 6.5

1. $\log\left|\frac{x+1}{x+2}\right|+c$

2. $\log\left|\frac{(x+2)^4}{(x+2)^2}\right|+c$

3. $\log\left|\frac{\sqrt{(x-1)(x-3)^3}}{(x-2)^2}\right|+c$

4. $\log\left|\frac{\sqrt{(x-1)(x-3)^9}}{(x-2)^4}\right|+c$

5. $\frac{1}{2}x+\log|x|-\frac{3}{4}\log|1-2x|+c$

6. $\log|x-1|-\frac{3}{x-1}-\frac{3}{2(x-1)^2}+c$

7. $3 \log |x - 2| - \dfrac{5}{(x-2)} + c$

8. $\dfrac{1}{4} \log |x| - \dfrac{1}{2x} + \dfrac{3}{4} \log |x+2| + c$

9. $\dfrac{2}{9} \log \left| \dfrac{x-1}{x+2} \right| - \dfrac{1}{3(x-1)} + c$

10. $\dfrac{1}{2} \log |1 + x^2| - \log |1 - x| + \tan^{-1} x + c$

11. $\dfrac{11}{4} \log \left| \dfrac{x+1}{x+3} \right| + \dfrac{5}{2(x+1)} + c$

12. $\dfrac{5}{3} \log |x+1| + \dfrac{5}{6} \log |x-2| - \dfrac{5}{2} \log |x+2| + c$

13. $\dfrac{2}{3} \log |x-1| - \dfrac{1}{3} \log |x^2 + x + 1| + \dfrac{2}{\sqrt{3}} \tan^{-1} \dfrac{2x+1}{\sqrt{3}} + c$

14. $\dfrac{x^2}{2} + 3x - \log |x-1| + \log |(x-2)^8| + c$

15. $\log \left| \dfrac{2 - \sin x}{1 - \sin x} \right| + c$

16. $\log |x| - \dfrac{1}{2} \log |x^2 + x + 1| - \dfrac{1}{\sqrt{3}} \tan^{-1} \dfrac{2x+1}{\sqrt{3}} + c$

17. $\dfrac{1}{2\sqrt{2}} \tan^{-1} \dfrac{x^2 - 1}{\sqrt{2}\, x} - \dfrac{1}{4\sqrt{2}} \log \left| \dfrac{x^2 - \sqrt{2}\, x + 1}{x^2 + \sqrt{2}\, x + 1} \right| + c$

18. $\dfrac{1}{2\sqrt{2}} \tan^{-1} \dfrac{x^2 - 4}{2\sqrt{2}x} + c$

19. $\dfrac{a^3 + ab + c}{(a-b)(a-c)} \log |x - a| + \dfrac{ab^2 + b^2 + c}{(b-a)(b-c)} \log |x - b|$

$+ \dfrac{ac^2 + bc + c}{(c-a)(c-b)} \log |x - c| + k$

20. $\dfrac{bp+c}{(p-q)(p-r)}\log|x-p|+\dfrac{bq+c}{(q-p)(q-r)}\log|x-q|+\dfrac{br+c}{(r-p)(r-q)}\log|x-r|+c$

21. $2\log|\sin^2\theta-4\sin\theta+5|+7\tan^{-1}(\sin\theta-2)+c$

22. $\dfrac{1}{6}\log|\tan^2\phi-\tan\phi+1|-\dfrac{1}{3}\log|\tan\phi+1|+\dfrac{1}{\sqrt{3}}\tan^{-1}\left(\dfrac{2\tan\phi-1}{\sqrt{3}}\right)+c$

23. $\dfrac{x^2}{2}+\log|x^2-1|+\dfrac{1}{2}\log\left|\dfrac{x-1}{x+1}\right|+c$

EXERCISE 6.6

1. $\dfrac{x^2}{4}(2\log|x|-1)+c$

2. $\dfrac{x^3}{9}(3\log|x|-1)+c$

3. $\dfrac{x^2}{4}(2\log|2x|-1)+c$

4. $x\sin x+\cos x+c$

5. $x\sin^{-1}x+\sqrt{1-x^2}+c$

6. $\dfrac{\sin^{-1}x}{4}(2x^2-1)+\dfrac{x\sqrt{1-x^2}}{4}+c$

7. $\dfrac{x^2}{3}\sin^{-1}x+\dfrac{1}{9}(2+x^2)\sqrt{1-x^2}+c$

8. $(x^2+1)\dfrac{\tan^{-1}x}{2}-\dfrac{x}{2}+c$

9. $\dfrac{\cos^{-1}x}{4}(2x^2-1)-\dfrac{x}{4}\sqrt{1-x^2}+c$

10. $(\sin^{-1}x)^2x+2\sqrt{1-x^2}\sin^{-1}x-2x+c$

12. $\dfrac{x^3}{9}(3\log|x|-1)+c$

13. $x\sin x+\cos x+c$

15. $\dfrac{e^{2x}}{5}(2\sin x-\cos x)+c$

16. $\dfrac{e^{ax}}{a^2+b^2}(a\cos bx+b\sin bx)+c$

17. $e^x\sin x+c$

18. $\dfrac{\sec\theta\tan\theta}{2}+\dfrac{1}{2}\log|\sec\theta+\tan\theta|+c$

EXERCISE 6.7

1. $2\sin^{-1}\frac{x}{2} + \frac{1}{2}x\sqrt{4-x^2} + c$ **2.** $\frac{1}{4}\sin^{-1}2x + \frac{1}{2}x\sqrt{1-4x^2} + c$

3. $\frac{1}{2}x\sqrt{4+x^2} + 2\log|x+\sqrt{4+x^2}| + c$ **4.** $\frac{x}{6}\sqrt{9+x^2} + \frac{3}{2}\log|x+\sqrt{9+x^2}| + c$

5. $\frac{2x+3}{4}\sqrt{x^2+3x} - \frac{9}{8}\log\left|x+\frac{3}{2}+\sqrt{x^2+3x}\right| + c$

6. $\frac{x+2}{2}\sqrt{x^2+4x+6} + \log|x+2+\sqrt{x^2+4x+6}| + c$

7. $\frac{x+2}{2}\sqrt{x^2+4x+1} - \frac{3}{2}\log|(x+2)+\sqrt{x^2+4x+1}| + c$

8. $\frac{5}{2}\sin^{-1}\left(\frac{x+2}{\sqrt{5}}\right) + \frac{x+2}{2}\sqrt{1-4x-x^2} + c$

9. $\frac{1}{3}(x^2+x)^{\frac{3}{2}} - \frac{(2x+1)\sqrt{x^2+x}}{8} + \frac{1}{16}\log\left|x+\frac{1}{2}+\sqrt{x^2+x}\right| + c$

10. $\sqrt{x^2-x-2} + \frac{1}{2}\log\left|x-\frac{1}{2}+\sqrt{x^2-x-2}\right| + c$

11. $2\sqrt{x^2-x-2} + \mathrm{c}$

12. $-3\sqrt{6+x-2x^2} + \frac{13}{2\sqrt{2}}\sin^{-1}\left(\frac{4x-1}{7}\right) + c$

13. $\frac{1}{\sqrt{3}}\log\left|\frac{\sqrt{3}+\tan\frac{\theta}{2}}{\sqrt{3}-\tan\frac{\theta}{2}}\right| + c$ **14.** $\frac{2}{\sqrt{3}}\tan^{-1}\left[\frac{1}{\sqrt{3}}\tan\frac{\theta}{2}\right] + \mathrm{c}$

15. $\frac{1}{\sqrt{3}} \log \left| \frac{\tan \frac{\theta}{2} - 2 - \sqrt{3}}{\tan \frac{\theta}{2} - 2 + \sqrt{3}} \right| + c$

16. $\frac{1}{\sqrt{15}} \log \left| \frac{\sqrt{3} + \sqrt{5} \tan \frac{\theta}{2}}{\sqrt{3} - \sqrt{5} \tan \frac{\theta}{2}} \right| + c$

17. $\frac{1}{12}$ [cosec $\theta - \cot \theta$] + c or $\frac{1}{12} \tan \frac{\theta}{2} + c$

18. $\frac{2}{\sqrt{a^2 - b^2}} \tan^{-1} \left[\frac{a \tan \frac{x}{2} + b}{\sqrt{a^2 - b^2}} \right] + c$ if $a > b$

or, $\frac{1}{\sqrt{b^2 - a^2}} \log \left| \frac{a \tan \frac{x}{2} + b - \sqrt{b^2 - a^2}}{a \tan \frac{x}{2} + b + \sqrt{b^2 - a^2}} \right| + c$ if $a < b$

EXERCISE 6.8

1. $x^3 e^x - 3x^2 e^x + 6xe^x - 6e^x + c$
2. $\frac{x^4 e^{2x}}{2} - x^3 e^{2x} + \frac{3}{2} x^2 e^{2x} - \frac{3}{2} xe^{2x} + \frac{3}{4} e^{2x} + c$
3. $-x^2 e^{-x} - 2xe^{-x} - 2e^{-x} + c$
4. $-x^3 e^{-x} - 3x^2 e^{-x} - 6xe^{-x} - 6e^{-x} + c$
5. $x^3 \sin x + 3x^2 \cos x - 6x \sin x - 6 \cos x + c$
6. $x^2 \sin x + 5x^4 \cos x - 20x^3 \sin x - 60x^2 \cos x + 120x \sin x + 120 \cos x + c$
7. $\frac{x^3}{2} \sin 2x + \frac{3}{4} x^2 \cos 2x - \frac{3x}{4} \sin 2x - \frac{3}{8} \cos 2x + c$
8. $\frac{x^4 \sin 2x}{2} + x^3 \cos 2x - \frac{3}{2} x^2 \sin 2x - \frac{3}{2} x \cos 2x + \frac{3}{4} \sin 2x + c$
9. $(x - 1)^2 e^x - 2e^x (x - 1) + 2e^x + c$
10. $\frac{(x - 1)^2 e^{2x}}{2} - \frac{(x - 1) e^{2x}}{2} + \frac{e^{2x}}{4} + c$
11. If $I_n = \int \cos^n x \, dx$ then $n I_n = \cos^{n-1} x \sin x + (n - 1) I_{n-2}$

12. If $I_n = \int \sin^n x \, dx$ then $n \, I_n = -\sin^{n-1} x \cos x + (n-1) I_{n-2}$

13. $\frac{3x}{128} - \frac{\sin 4x}{128} + \frac{\sin 8x}{1024} + c$

14. $\frac{1}{5}\sin^5 x - \frac{1}{7}\sin^7 x + c$

15. $\frac{1}{7}\cos^7 x - \frac{1}{5}\cos^5 x + c$

16. $\frac{5}{128}x + \frac{\sin 2x}{64} - \frac{\sin 4x}{128} - \frac{\sin 6x}{192} - \frac{\sin 8x}{1024} + c$

17. $\frac{1}{13}\cos^{13} x - \frac{1}{11}\cos^{11} x + c$

MISCELLANEOUS EXERCISE ON CHAPTER 6

1. $x \sin^{-1} 2x + \frac{1}{2}\sqrt{1-4x^2} + c$

2. $\frac{1}{3}\tan^3 x - \tan x + x + c$

3. $\frac{1}{2}\tan^2 x + \log|\cos x| + c$

4. $-\cot x - x + c$

5. $\frac{3}{8}x + \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + c$

6. $\frac{1}{5}\log\left|\frac{5 - 3\sin x + 4\cos x}{3\cos x + 4\sin x}\right| + c$

7. $2\sqrt{e^x - 1} - 2\tan^{-1}(\sqrt{e^x - 1}) + c$

8. $\frac{2}{5}(x-1)^{\frac{5}{2}} + \frac{4}{3}(x-1)^{\frac{3}{2}} + 2\sqrt{x-1} + c$

9. $\frac{2}{3}(x-1)^{\frac{3}{2}} + 4\sqrt{x-1} - \frac{2}{\sqrt{x-1}} + c$

10. $\frac{1}{2}\log\left|\frac{x-\sqrt{x^2+1}}{x+\sqrt{x^2+1}}\right| - \frac{1}{\sqrt{2}}\log\left|\frac{\sqrt{2}\,x-\sqrt{x^2+1}}{\sqrt{2}\,x+\sqrt{x^2+1}}\right| + c$

11. $\frac{1}{8}\log\left|\frac{1-\sin x}{1+\sin x}\right| - \frac{1}{4\sqrt{2}}\log\left|\frac{1-\sqrt{2}\sin x}{1+\sqrt{2}\sin x}\right| + c$

12. $-\frac{\sqrt{1+x^2}}{x} + c$

13. $\sqrt{x^2+ax} + a\log|\sqrt{x} + \sqrt{a+x}| + c$

14. $-\frac{1}{\sqrt{5}}\log\left|\frac{4+x+\sqrt{5x^2+20}}{5(x-1)}\right| + c$

15. $\sin^{-1}\left[\frac{1}{2}\sin x\right] + c$

16. $-\frac{1}{2}\log|x+1| + \frac{1}{4}\log|x^2+9| + \frac{3}{2}\tan^{-1}\left(\frac{x}{3}\right) + c$

17. $\log|x| + c$

18. $-\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x + c$

19. $x\cos\alpha + \sin\alpha\log|\sin(x-\alpha)| + c$

20. $\frac{1}{\sqrt{2}}\log|\cos x + \sin x + \sqrt{\sin 2x}| - \frac{1}{\sqrt{2}}\sin^{-1}(\cos x - \sin x) + c$

21. $e^x - \frac{2e^x}{x+1} + c$

22. $e^x\tan x + c$

23. $\frac{1}{4}\log\left|\frac{x^2+x+1}{x^2-x+1}\right| + \frac{1}{2\sqrt{3}}\tan^{-1}\left[\frac{x^2-1}{\sqrt{3}x}\right] + c$

24. $-2\log|x+1| - \frac{1}{x+1} + 3\log|x+2| + c$

25. $\sqrt{\frac{x-1}{x+1}} + c$

26. $\frac{1}{6}\log|1-y|+\frac{1}{2}\log|1+y|-\frac{2}{3}\log|1+2y|+c$, where $y=\cos x$

27. $\frac{1}{10}\log|1-\cos x|-\frac{1}{2}\log|1+\cos x|+\frac{2}{5}\log|3+2\cos x|+c$

28. $x\tan x+\log|\cos x|-x\sec x+\log|\sec x+\tan x|+c$

29. $-\frac{1}{4}\tan^{-1}\left[\frac{1}{2}\cot x\right]+c$

30. $-\frac{1}{5}e^{2x}\cos x+\frac{2}{5}e^{2x}\sin x+c$

31. $-\frac{1}{4}(1-2x^2)\sin^{-1}x+\frac{1}{4}x\sqrt{1-x^2}+c$

32. $3x\tan^{-1}x-3\log[\sqrt{1+x^2}]+c$

33. $\frac{1}{4}\tan^{-1}\left[\frac{x^2-1}{\sqrt{2}x}\right]-\frac{1}{4\sqrt{2}}\log\left|\frac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1}\right|+c$

34. $\log\left|\frac{2x-\alpha-\beta}{\alpha-\beta}+\sqrt{\left(\frac{2x-\alpha-\beta}{\alpha-\beta}\right)^2-1}\right|+c$

35. $\tan^{-1}\left(1+\tan\frac{x}{2}\right)+c$

36. $\frac{2(2x-1)}{\pi}\sin^{-1}\sqrt{x}+\frac{2\sqrt{x}\sqrt{1-x}}{\pi}-x+c$

37. $x\log(\log x)-\frac{x}{\log x}+c$

38. $-2\sqrt{1-x}+\cos^{-1}\sqrt{x}+\sqrt{x}\sqrt{1-x}+c$

39. $\frac{-1}{4}\cos(\tan^{-1}x^4)+c$

40. $\log\left|\frac{1+2\log x}{2+3\log x}\right|+c$

41. $\dfrac{-2}{\sin\,\alpha}\sqrt{\dfrac{\sin\,(x+\alpha)}{\sin x}}+c$

42. $\sqrt{2}\,\tan^{-1}\left[\dfrac{\tan x-1}{\sqrt{2\tan x}}\right]+c$

EXERCISE 7.1

2. (a) $\dfrac{35}{2}$ (b) $\dfrac{15}{2}$ (c) $\dfrac{27}{2}$ (d) $e-\dfrac{1}{e}$

3. (a) 6 (b) $\dfrac{9}{2}$ (c) -4 (d) $184\,\dfrac{1}{2}$

EXERCISE 7. 2

1. 1 **2.** 0 **3.** 0 **4.** 1

5. 1 **6.** 2 **7.** $1-\dfrac{1}{\sqrt{2}}$ **8.** $e^4\,(e-1)$

9. 1 **10.** 2 **11.** 0 **12.** $\dfrac{3}{2}$

13. 2 **14.** $\log\dfrac{3}{2}$ **15.** $\dfrac{64}{3}$ **16.** $\dfrac{1}{2}\log 2$

17. $\dfrac{1}{2}\log 2$ **18.** $\log(\sqrt{2}+1)$ **19.** $\log(\sqrt{2}-1)(2+\sqrt{3})$

20. $\dfrac{\pi}{2}$ **21.** $\dfrac{\pi}{4}$ **22.** $\dfrac{1}{2}\log\dfrac{3}{2}$ **23.** $\dfrac{\pi}{4}$

24. $\dfrac{1}{2}\log 2$ **25.** $\dfrac{1}{5}\log 6+\dfrac{3}{\sqrt{5}}\tan^{-1}\sqrt{5}$ **26.** $\dfrac{1}{4}\log\dfrac{5}{3}$

27. $\dfrac{e-1}{2}$ **28.** $\dfrac{64}{231}$ **29.** $\log\dfrac{9}{8}$

30. $5 - \frac{5}{2}\left(9\log\frac{5}{4} - \log\frac{3}{2}\right)$ **38.** 2 **39.** $a = -1, b = 1$ **40.** $\frac{9}{2}, \frac{1}{2}$

EXERCISE 7.3

1. $\frac{1}{2}\log 2$ **2.** $\frac{1}{5}\log 6 + \frac{3}{\sqrt{5}}\tan^{-1}\sqrt{5}$ **3.** $\frac{e-1}{2}$

4. $\frac{64}{231}$ **5.** $\frac{\pi}{2} - \log 2$ **6.** $\frac{16}{15}(2+\sqrt{2})$ **7.** $\frac{\pi}{2}$

8. $\frac{1}{\sqrt{5}}\log\left(\frac{3+\sqrt{5}}{2}\right)$ **9.** $\frac{1}{\sqrt{a^2+b^2}}\log\left|\frac{(a+\sqrt{a^2+b^2})\,a}{(\sqrt{a^2+b^2}-b)\,b}\right|$

10. $\frac{2}{3}\tan^{-1}\frac{1}{3}$ **11.** $\frac{1}{\sqrt{17}}\log\frac{21+5\sqrt{17}}{4}$

12. $\frac{1}{\sqrt{5}a}\log\left|\frac{7+3\sqrt{5}}{2}\right|$ **13.** 2

EXERCISE 7.4

1. $\frac{\pi}{4}$ **2.** $\frac{\pi}{4}$ **3.** $\frac{\pi}{8}\log 2$ **4.** $\frac{1}{(n+1)(n+2)}$

5. $\frac{\pi}{4}\log\frac{1}{4}$ **6.** 2 **7.** 0 **8.** $\frac{\pi}{2}$

9. π **10.** $\frac{\pi}{2\sqrt{2}}\log(1+\sqrt{2})$ **11.** 0 **12.** 9

13. 62 **14.** $3 - \frac{\pi}{2} + e^6$ **15.** 0 **22.** $\frac{\pi}{4}$

23. $\frac{\pi}{4}$ **24.** $\frac{\pi}{2} - 1$ **25.** $1 - \log 2$

26. $\frac{1}{40}\log 9$ **27.** $\frac{3\pi+1}{\pi^2}$

EXERCISE 7.5

1. $\frac{1}{6}$

2. $\frac{1}{3}$

3. $\frac{\pi}{4} - \frac{1}{2}$

4. 8

5. πab

6. $8\sqrt{3}$

7. $\frac{9\pi}{8} - \frac{9}{4} \sin^{-1} \frac{1}{3} + \frac{\sqrt{2}}{6}$

8. $\frac{15}{2}$

9. $\frac{8a^2}{3m^3}$

10. $\frac{1}{3}$

11. 4

12. $\frac{2\pi}{3} - \frac{\sqrt{3}}{2}$

13. $\frac{4}{3}$

14. $\frac{8}{3} a^2$

15. $\frac{9}{8}$ sq units

16. $\frac{ab}{4}(\pi - 2)$

EXERCISE 8.1

1. order 1, degree 1, nonlinear

2. order 1, degree 1, nonlinear

3. order 2, degree 1, nonlinear

4. order 2, degree 1, nonlinear

5. order 1, degree 2, nonlinear

6. order 1, degree 1, nonlinear

7. order 1, degree 2, nonlinear

8. order 2, degree 1, linear

9. order 2, degree 1, linear

10. order 2, degree 2, nonlinear

EXERCISE 8.2

1. $(x^2 - 2y^2)\, p^2 - 4pxy - x^2 = 0,\ p = \frac{dy}{dx}$

2. $(1 + p^2)^3 = r^2 \left(\frac{d^2 y}{dx^2} \right)^2 p = \frac{dy}{dx}$

3. $xy \frac{d^2 y}{dx^2} + x \left(\frac{dy}{dx} \right)^2 - y \frac{dy}{dx} = 0$

4. $(x - y)^2 (1 + y'^2) = (x + yy')^2$

5. $x^2(1+y'^2) = (x+yy')^2$

6. $x^2 - y^2 + 2x\,yy' = 0$

EXERCISE 8.4

1. (i) $\dfrac{x^6}{6} + \dfrac{1}{3}x^3 - 2\log|x| + c$ (ii) $y = 2\tan\dfrac{x}{2} - x + c$

(iii) $y = \dfrac{\sin^5 x}{5} - \dfrac{\sin^7 x}{7} - \dfrac{1}{6}(2x+1)^{\frac{3}{2}} + \dfrac{1}{10}(2x+1)^{\frac{5}{2}} + c$

(iv) $y = \dfrac{1}{2}x^2 + 2x - 13\log|x+2| + c$

(v) $y = \log(e^x + e^{-x}) + c$ (vi) $\dfrac{1}{x} + \dfrac{1}{y} + \log y - x = c$

(vii) $\tan x \tan y = c$

(viii) $y = c(x-1)^2 \exp\left(\dfrac{2}{3}x^3 + x^2 + 2x\right)$

(ix) $y = \tan\left(\dfrac{x^2}{2} + x + c\right)$ (x) $y = c\log x$

(xi) $\sin^{-1} y + \sin^{-1} x = c$ (xii) $y = c(a+x)(1-ay)$

(xiii) $\tan y = c(1-e^x)^3$ (xiv) $y = \dfrac{1}{3}e^{3x} - x^2 + c$

(xv) $\cos y + \dfrac{1}{3}\cos^3 x - \cos x = c$

2. (i) $y = \dfrac{2}{\sqrt{\cos 2x}}$ (ii) $y = 4x^{\frac{3}{2}}$

(iii) $y^2 = \dfrac{1}{8-4e^x}$ (iv) $y = \sin^{-1} x$

(v) $y - 2\log(y+2) = \log\left(\dfrac{x}{8}\right)$

EXERCISE 8.5

1. $y = cx + x \log x$

2. $\tan^{-1}\left(\frac{y}{x}\right) = c + \log x$

3. $x^2 y^2 + 2x^3 y = c$

4. $-\frac{2x}{x+y} = \log c\,(x+y)$

5. $x^2 + y^2 = cx^3$

EXERCISE 8.6

1. (i) $y = ce^{-2x} + 2x - 1$

(ii) $y = ce^{-ax} + \frac{e^{mx}}{m+a}$ if $m \neq -a$; $y = (c+x)\,e^{mx}$ if $m = -a$

(iii) $y = ce^{-x} + \frac{1}{2}(\cos x + \sin x)$

(iv) $y = ce^{-2x} + 2e^{x}$

(v) $y = ce^{2x} - \frac{1}{13}(2\cos 3x - 3\sin 3x)$

(vi) $y = ce^{-x} + \frac{1}{2}(\sin x - \cos x)$

(vii) $y = kx + cx^3$

(viii) $y = e^x + cx$

(ix) $y = \cos x + ce^{-x}$

(x) $y = x^2 + c\cos x$

(xi) $cx^2 y^2 + 2xy^2 - 1 = 0$

(xii) $y = \sin x + \frac{c}{x}$

(xiii) $4xy = y^4 + c$

(xiv) $4xy = 2x^2 \log x - x^2 + c$

(xv) $2y \log x = (\log x)^2 + c$

(xvi) $x = 2y^2 + cy$

2. $\frac{T-S}{150-S} = e^{-kt}$

3. (i) $i = ce^{\frac{R}{L}t}$ (ii) $i = \frac{E}{R} + ce^{\frac{-R}{L}t}$

4. $r = 3 - t\ mm,\ 0 < t < 3$

5. $(x-2)^2 + y^2 = 9$

6. $y = 0.01e^{-\left[\frac{\log 3}{10}\right]t}$

7. $t = \dfrac{\log 100}{\log 2}$ hours

8. 8.33 %

9. 20 log 2 years

10. $\dfrac{dp}{dt} = k(w - s)$, where $w = ap + b$.

EXERCISE 9.1

1. $x_2 - x_1$

2. $2\sqrt{2}\,\vec{i}$ where $\vec{i}$ is the unit vector along x-axis.

EXERCISE 9.2

1. $m = 2$

2. $n = \pm 12$

4. (i) $(2, -1)$ (ii) $(-1, -2)$

5. (i) $(1, -3)$ (ii) $(-4, -3)$

6. $(0, 2)$

12. $\sqrt{26}$

EXERCISE 9.3

1. -15 units

2. 40 units

4. 9 units

9. $\cos^{-1} \dfrac{10}{\sqrt{6}\,\sqrt{17}}$

EXERCISE 9.4

2. $6\sqrt{5}$ sq. units

5. $\vec{\beta_1} = \dfrac{3}{2}\vec{i} - \dfrac{1}{2}\vec{j},\quad \vec{\beta_2} = \dfrac{1}{2}\vec{i} + \dfrac{3}{2}\vec{j} - 3\vec{k}$

EXERCISE 9.5

1. $-\vec{i} - 3\vec{j} - 3\vec{k}$

2. $14\vec{i} - 8\vec{j} + 12\vec{k}$

3. $-3\vec{i} + \vec{j} + \vec{k}$

4. $\vec{i} - 11\vec{j} - 5\vec{k}$

5. 0

6. $\vec{i} - 11\vec{j} - 5\vec{k}$

MISCELLANEOUS EXERCISE ON CHAPTER 9

3. $(a_1 + b_1, a_2 + b_2, a_3 + b_3)$

4. (a) (i) $\left(\frac{4}{3}, \frac{7}{3}, \frac{17}{3}\right)$ (ii) $\left(-\frac{1}{3}, \frac{11}{3}, \frac{13}{3}\right)$

(b) (i) $\left(\frac{1}{3}, \frac{2}{3}, 1\right)$ (ii) $\left(\frac{2}{3}, \frac{1}{3}, 1\right)$

8. No **9.** (i) $\frac{3}{7}\vec{i} - \frac{6}{7}\vec{j} + \frac{2}{7}\vec{k}$ (ii) $\frac{12}{13}\vec{i} - \frac{5}{13}\vec{k}$

13. (a) $\frac{16}{49}\vec{i} + \frac{48}{49}\vec{j} + \frac{24}{49}\vec{k}$ (b) $\frac{434}{59}\vec{i} + \frac{62}{59}\vec{j} - \frac{186}{59}\vec{k}$

14. (a) $-\frac{24}{63}$ **15.** $-\vec{i} + 11\vec{j} - 2\vec{k}$ **27.** $3\sqrt{5}$ sq. units

31. 3 **32.** $x - 2y - z + 1 = 0$

EXERCISE 10.1

1. $\left(\frac{20}{7}, \frac{5}{7}, \frac{15}{7}\right)$ **2.** $2, -2, -1; \frac{2}{3}, -\frac{2}{3}, \frac{1}{3}$

3. $\frac{2}{7}, \frac{-3}{7}, \frac{6}{7}; 6\vec{i}, -9\vec{j}, 18\vec{k}$ **4.** $\frac{1}{\sqrt{14}}, -\frac{3}{\sqrt{14}}, \frac{2}{\sqrt{14}}$

5. (a) $\frac{2}{3}, \frac{2}{3}, \frac{-1}{3}$; (b) $\frac{6}{7}, \frac{-2}{7}, \frac{-3}{7}$; (c) $\frac{3}{5}, 0, \frac{4}{5}$

6. (a) $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right), \cos^{-1}\left(\frac{1}{\sqrt{3}}\right), \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$

(b) $\cos^{-1}\left(\frac{4}{9}\right), \cos^{-1}\left(\frac{8}{9}\right), \cos^{-1}\left(\frac{1}{9}\right)$

(c) $\frac{\pi}{2}, \frac{\pi}{4}, \frac{3\pi}{4}$

7. $2\sqrt{3}(\vec{i}+\vec{j}+\vec{k})$ 8. (a) $\frac{5}{2}\vec{i}-\frac{7}{4}\vec{k}$; (b) $-5\vec{j}-\frac{1}{2}\vec{k}$

9. $\left(\frac{5}{3}, \frac{7}{3}, \frac{17}{3}\right)$ 10. $(4, 5, -5)$ 11. $\frac{36}{65}$

EXERCISE 10 .2

1. $\vec{r} = 5\vec{i} - 4\vec{j} + 6\vec{k} + \lambda(3\vec{i} + 7\vec{j} + 2\vec{k})$

2. $\vec{r} = 2\vec{i} - \vec{j} + 4\vec{k} + \lambda(\vec{i} + \vec{j} - 2\vec{k})$; $x - 2 = y + 1 = \frac{z-4}{-2}$

3. $\vec{r} = 3\vec{i} + 4\vec{j} - 7\vec{k} + \lambda(-2\vec{i} - 5\vec{j} + 13\vec{k})$

4. $\left(\frac{13}{5}, \frac{23}{5}, 0\right)$

5. $\vec{r} = 4\vec{i} + 5\vec{j} + 10\vec{k} + \lambda(\vec{i} + \vec{j} + 3\vec{k})$; $4 - x = 5 - y = \frac{10-z}{3}$
$\vec{r} = 2\vec{i} + 3\vec{j} + 4\vec{k} + \lambda(\vec{i} + \vec{j} + 5\vec{k})$

6. $1 : 2 : 3$, $\vec{r} = 2\vec{i} - \vec{j} - \vec{k} + \lambda(\vec{i} + 2\vec{j} + 3\vec{k})$

7. (a) $\cos^{-1}\left(\frac{19}{21}\right)$ (b) $\cos^{-1}\left(\frac{2}{3}\right)$ (c) $\cos^{-1}\left(\frac{8\sqrt{3}}{15}\right)$

8. (a) $\cos^{-1}\left(\frac{3}{\sqrt{182}}\right)$ (b) 0 9. (a) $\cos^{-1}\left(\frac{1}{65}\right)$ (b) 60^0

10. $\frac{10}{\sqrt{59}}$ 11. $\frac{3\sqrt{2}}{2}$ 12. $\frac{8}{\sqrt{29}}$

13. (a) No (b) Yes (c) No

EXERCISE 10.3

1. $\frac{10}{\sqrt{27}}$ 3. $\vec{r} = 3\vec{i} + \vec{j} + 2\vec{k} + \lambda(2\vec{i} - \vec{j} + \vec{k})$; $\left(2, \frac{3}{2}, \frac{3}{2}\right)$

4. (a) $\vec{r}\cdot(5\vec{i} - 2\vec{j} - 3\vec{k}) = 7$; (b) $\vec{r}\cdot(\vec{j} - 2\vec{k}) = 2$; (c) $\vec{r}\cdot(2\vec{i} + \vec{k}) = 5$

5. (a) $x + y - z = 2$; (b) $2x + 3y - 4z = 1$; (c) $2x - 5y - z = -15$

6. $\vec{r}\cdot(\vec{i}-\vec{j}-\vec{k})=0$

7. $\vec{r}=\lambda(\vec{i}-2\vec{j}+\vec{k})$

8. $\frac{1}{\sqrt{14}}\vec{i}+\frac{2}{\sqrt{14}}\vec{j}+\frac{3}{\sqrt{14}}\vec{k}$

9. $\cos^{-1}\left(\frac{-5}{\sqrt{29}\sqrt{2}}\right)$

10. $-2x + y - 3z = 8$

11. $\frac{13}{\sqrt{21}}$

12. $23x - 13y + 32z + 45 = 0$, $5x - y - 4z - 3 = 0$

13. $x + y + z = 0$

EXERCISE 10.4

1. $x^2 + y^2 + z^2 - 6x + 2y - 4z - 2 = 0$

2. $x^2 + y^2 + z^2 - 2x - 2y - 2z + 2 = 0$

3. $(3, -4, 5)$; 7

4. (a) $\left(\frac{1}{2}, -1, -\frac{1}{2}\right)$; 1 (b) $\left(\frac{a}{2}, \frac{b}{2}, \frac{c}{2}\right)$; $\frac{1}{2}\sqrt{a^2+b^2+c^2}$

5. (a) $x^2 + y^2 + z^2 - 8x - 6y - 8z + 38 = 0$. Centre $(4, 3, 4)$, $r = \sqrt{3}$

   (b) $x^2 + y^2 + z^2 - 3x + 4y - 2z - 5 = 0$, Centre $\left(\frac{3}{2}, -2, 1\right)$; $r = \frac{7}{2}$

6. $x^2 + y^2 + z^2 + 3(x - y + z) - 56 = 0$

10. (a) $x^2 + y^2 + z^2 - 3x + 5y + 7 = 0$

    (b) $x^2 + y^2 + z^2 + 3(x - y + z) - 56 = 0$

$$\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$$

$$\vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

1260

MATHEMATICS PART II